The Pricing of VWAP Options under a Lévy Process Framework

Leanne J.Dong

Supervised by Professor Alexander Novikov

Thesis submitted to the School of Mathematical Sciences
in partial fulfilment of requirements for Bachelor of Mathematics
(Honours)

University of Technology, Sydney
February 2012

Abstract

In this thesis, we aim to derive analytical formulae to price the so-called Volume Weighted Average Price (VWAP) options. In particular, the stock price is assumed to evolve as a geometric Lévy process and the trade volume process is modelled via a shifted squared Ornstein Uhlenbeck process. First, the theory of Lévy processes is discussed from both a probabilistic and stochastic analysis point of view; Next, analytical formulae for the first two moments of VWAP are derived, numerically computed and VWAP call options prices are found; Finally, VWAP is simulated to benchmark the analytical results.

Acknowledgements

I would like to express my deepest appreciation to my supervisor, Professor Alexander Novikov, for his advice and support in completing my thesis. Without his guidance, this thesis would not have been possible. I always consider myself extremely lucky having the chance to work with and learn from him.

I would like to thank Mr Tim Ling for providing the Mathematica notebook and the C++ codes for the previous model. These codes were essential in constructing a new model in this thesis.

I also owe my gratitude to Dr Yakov Zinder, who is the director of the Graduate Diploma in Mathematics and the former coordinator of the Honour program, without his support I would not been able to transit from Finance to Mathematic.

In addition, I thank all members of the School of Mathematical Science, including coordinators of the Honours year and lecturers who taught the Mathematics subjects in the Honour year, Graduate Certificate and Graduate Diploma year.

Lastly, I thank those who spent time reading parts of this thesis and provided helpful writing advice.

Leanne Dong, January 2012.

Contents

List of Figures

List of Tables

Notation and Symbol Description

X_t or $X(t)$: The value of a stochastic process at time t
X_t^{VG}: Variance Gamma Process
$\psi(\cdot)$: Characteristic exponent
$\varphi(\cdot)$: Characteristic function
$\ll$: Absolute continuity
$\gamma_t^{(\mu,\nu)}$: Gamma process with mean rate μ per unit of time and variance ν
$\gamma_t^{(\nu)}$: Gamma process with mean rate 1 per unit of time and variance ν
$\tau_\gamma(t) := \Gamma(\frac{t}{\nu}, \nu)$: Time index (stopping time) that is a Gamma process with shape parameter $\frac{t}{\nu}$ and scale parameter ν
$\tau_{\mathrm{IG}}(t)$: Time index (stopping time) that follows a Inverse Gaussian process
$\tau_{\mathrm{GIG}}(t)$: Time index (stopping time) that follows a Generalised Inverse Gaussian process
$\varphi_\gamma(\cdot, t)$: Characteristic function of gamma process X_t^G
$B_{\tau_\gamma}^{(\theta,\sigma)}$: Lévy subordinated (time-changed) Brownian motion with independent Gamma subordinator
$B_{\tau_{\mathrm{NIG}}}^{(\theta,\sigma)}$: Lévy subordinated (time-changed) Brownian motion with independent Inverse Gaussian subordinator
$B_{\tau_{\mathrm{GH}}}^{(\theta,\sigma)}$: Lévy subordinated (time-changed) Brownian motion with independent Generalised Inverse Gaussian subordinator
$\Gamma(x)$: Gamma function
$\mathbb{R}_0$: $\mathbb{R}\backslash\{0\}$
$\mathcal{B}$: The class of Borel sigma algebras;
$\mathcal{B}(\cdot)$: the set of Borel sigma algebras restricted to domain $\mathbb{R}_0$;
Ω: The set of scenario of randomness
$N(\cdot)$: Jump measure
$\Phi(\cdot)$: Cumulative distribution function of the standard normal distribution
$C^{2,1}$: A continuous function that is twice differentiable in the first variable and differentiable in the second variable.
$\mathbb{F}$: $\{\mathcal{F}_t, t \in [0, T]\}$
π: Economic profit (Microeconomics)
ζ: Trading strategy
$\mathcal{L}$: Generator (PDE)
$\mathbb{E}(\cdot)$: Expectation operator under physical measure (real world measure) $\mathbb{P}$
$\mathbb{E}^{\mathbb{Q}}(\cdot)$: Expectation operator under risk neutral measure $\mathbb{Q}$
$K_p(\cdot)$: Modified Bessel function of the second kind with index p

Remarks on notation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}^+$ are, respectively, the collections of positive integers, all integers, all rational numbers, all real numbers and all strictly positive real numbers.

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space: Ω is the sample space, $\mathcal{F}$ is the sigma algebra, $\mathbb{P}$ is the probability measure.
In this thesis, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, the filtration is denoted by $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$. $\mathcal{F}_0$ is assumed to be deterministic and all processes considered in this thesis are adapted to $\mathbb{F}$.

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ that defined on the same measurable space $(\Omega, \mathcal{F})$ are introduced.

Càdlàg means right continuous with left limit. The function f is càdlàg if $f(t) = f(t+)$ It is chosen to model discontinuous process whose values are unpredictable. Trajectories of processes with jumps are described by càdlàg function. In this thesis, this word is used whenever we speak of an unanticipated process or discontinuous process whose values are unpredictable. One main example throughout this thesis is the trajectories of stock price. On the other hand, càglàd means left continuous with right limit. The function f is càglàd if $f(t) = f(t-)$. It is chosen to model a discontinuous process whose values are predictable. This will be the case when we speak of the modelling of trading strategies via a stochastic integral in Chapter 2.

δ_x is a measure given by $\delta_x(A) = \mathbb{I}\{x \in A\} = \mathbb{I}_A(x)$ for every Borel set A.

For a semimatingale X, $[X,X]$ denotes its quadratic variation and $[X,X]^c$ is the continuous part.

List of Abbreviation

CLT= Central Limit Theorem

VG= Variance Gamma

NIG= Normal Inverse Gaussian

GIG= Generalised Inverse Gaussian

VWAP=Volume-Weighted Average Price

r.v.=Random Variable

w.r.t.=With Respect To

a.s.=Almost Surely

i.i.d.=Independent Identically Distributed

SDE=Stochastic Differential Equation

ODE=Ordinary Differential Equation

PDE=Partial Differential Equation

BS=Black-Scholes

pdf=Probability Density Function

GBM= Geometric Brownian motion

GL= Geometric Lévy

E.M.M. = Equivalent Martingale Measure

LDS = Low Discrepancy Sequences

Chapter 1

Introduction

1.1 Motivation and Literature

VWAP options are derivatives written on VWAP. VWAP stands for Volume Weighted Average Price. These derivative products are relatively new to the market place. An important question that our thesis addresses is how to price such derivatives.

Most of the existing literature on VWAP focuses on the development of trading strategies and algorithms [62]. We are aware of only two papers and one dissertation that discuss VWAP from an option pricing point of view [62; 79; 80]. The two papers and the dissertation utilise a technique called moment matching. In our view, it is crucial to analyse each method and to identify a suitable method to work with. The main contribution of the paper [80] is the development of a moment-matching method to find a lognormal approximation for the call option via approximating VWAP first and second moments using a truncated Taylor series expansion. The author worked under a continuous time setting for VWAP with a geometric Brownian motion for $S(t)$ and a Cox-Ingersoll-Ross (CIR) model for volume process $U(t)$. It was shown that approximation for the first and second moments of VWAP can be found by solving a large (19 equations!) system of ODEs. The author in paper [80] has also contributed the following in his Ph.D. thesis.

- Derivation of a fundamental pricing PDE that describes the price of the VWAP options;
- Derivation of analytical formulae of the option bounds for both fixed and float strike VWAP options;
- Evaluation of VWAP options Greeks via Finite Difference, Pathwise and Likelihood ratio methods;
- The paper [80] and the methodology in pricing VWAP exotic options (VWAP digital options);
- Solving the fundamental pricing PDE via Finite Difference, Crank-Nicolson and Alternating Direction Implicit schemes;
- Development of a series solution to the VWAP option price.

The Novikov et al. [62] paper studies the approach in paper [80] and develops a semi-analytical method. Three improvements are made. The first improvement is the approximation method used in computing the moments of VWAP that involved a ratio of two integrals. In the former, the approximation method used required solving a large (19 equations!) system of ODEs. In contrast, in the latter, the two moments are approximated by calculating the Laplace Transform of the integral of the squared Ornstein-Uhlenbeck process. It is shown that calculation of this type relies only on change of measure via Girsanov theorem and does not involve solving PDEs and ODEs. This idea comes from the context of calibration of an Ornstein-Uhlenbeck process [61]. Another improvement in the latter approach is the derivation of the **exact** analytical formulae for the moments of VWAP. The third improvement attributed to the choice of distribution being matched, the moments of VWAP are matched to the Generalised Inverse Gaussian (GIG) distributed in addition to matching with the lognormal distributed. The numerical results comparison indicates that the GIG approximation is more accurate than the lognormal approximation. Nevertheless, we will detail both approaches in chapter 3 in more depth.

As one can see, Stace's approach works, but not as efficient. Solving a system of 19 ODEs is very computational intensive and mistakes are more likely to occur. It was identified by one of the author in paper [62] that one of the solution of the system of the 19 ODEs was in fact false. Hence, we adopt the semi-analytical approach developed by Novikov et al. (2010) [62] as a main reference for this thesis due to its simplicity and accuracy.

However, we would like to relax the following assumption on the original work of the original work of [62].

- Stock price $S(t)$ evolves as a GBM, i.e. $S(t) \sim \text{lognormal}(\mu - \frac{1}{2}\sigma^2, \sigma^2)$
- The Brownian motion under the price dynamics is uncorrelated with the Brownian motion in the volume dynamics, which leads to the independence between the model for the stock price $S(t)$ and the volume $U(t)$.

Consider the first assumption, recall the classical diffusion model (Merton (1973), Black and Scholes (1973)) for the process $S(t)$ is

$$dS(t) = S(t)(\mu dt + \sigma dW(t))$$

where $W(t)$ is a standard Wiener process, μ is the expected return and σ is the stock price volatility. The solution of this equation is

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \tag{1.1}$$

The increments of the Log price are called continuous compounded return and are then given as

$$L(t) = \log(S(t)) - \log(S(t-1)) = (\mu - \frac{1}{2}\sigma^2)t + \sigma(W(t) - W(t-1))$$

which implies that returns are independent identically distributed (i.i.d.) Gaussian random variables. It is well known that contrary to a Wiener process, $\ln(S(t))$ are neither Gaussian, nor homogeneous and do not possess the property of independent increments (see [56]). Indeed, the distribution of log returns appear to be fat-tailed and asymmetrical, and these properties are not explained in the diffusion based (Black Scholes) model. There is another important property of the market that is not explained in Black Scholes, which is the phenomenon "volatility smile". This is caused by the assumption that volatility is constant over time in the Black Scholes model. Let us now analyse this problem.

1.1.1 Evidence of inadequacy of Brownian motion: Volatility skew and smiles

The validity of an option pricing model is justified by the ability of capturing the state of the options market at a given instant. A valid model for volatility smile should be capable of calibrating to liquid stocks, bonds and options, then can be used to interpolate to the hidden hedging ratio (Delta) of a regular or an exotic option. One of the well known smile model is the Black Scholes, it translates market price of the option into a expression in function of implied volatility, denoted as $\hat{\sigma}_t(T, K)$. Considering the plain vanilla European call with payout

$$(S(T) - K)^+$$

Practically, such a option is expressed in term of two variables: the moneyness $M = \frac{K}{S(0)}$ which is the extent to which the option is in or out-of-the-money, and the time to maturity $\tau = T - t$. At any time t, the Black Scholes gives

$$C_i^{BS} = f_i^{BS}(t, M, \tau, \hat{\sigma}) = S(0)\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \tag{1.2}$$

where

$$d_{1,2} = \frac{-\ln(M) + \tau(r \pm \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}}; \quad M = \frac{K}{S(0)}; \quad \tau = T - t \ \ ; \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx$$

Since equation (1.2) is a continuous function of σ, mapping $(0, \infty)$ into $((S(t) - Ke^{-r\tau})^+, S(t))$[1]. Hence, given any market price $C_t^*(T_i, K)$ of a call option matures at T_i, one can always infer the value of $\hat{\sigma}_t(T, K)$ to match the Black Scholes price with market price. Mathematically,

$$\exists! \ \ \hat{\sigma}_t(T_i, K) > 0 : \quad f_i^{BS}(t, M, \tau, \hat{\sigma}_t(T_i, K)) = C_t^*(T_i, K)$$

$\hat{\sigma}_t(T_i, K)$ is called (BS) implied volatility of the option, denoted as $\hat{\sigma}_t^{BS}(T_i, K)$ from now. If the assumptions underlying the Black Scholes formula were correct, the plot of the implied volatility of an option across all level of strike prices and time to maturities should look flat, because the assumption of the Black Scholes model is that the implied volatility $\hat{\sigma}_t$ should not depend on i and the moneyness $\frac{K}{S(0)}$.

Some possible causes of the volatility smile could be:

- Insufficient number of parameters in accurately describing the distribution of interest. In Black Scholes model, stock price is assumed to evolve as a geometric Brownian motion with only two parameters (M, σ^2). Empirically, asset price are fat-tailed and skewed, hence more parameters need to be considered. However, an increase of parameters results in a loss of model parsimony and a balance need to be struck.

- Observed price processes are not continuous, they exhibit jumps and spikes.

In spite of these mentioned shortcomings, the Black Scholes model that based on Brownian motion remains a reference model due to its simplicity. To preserve the parsimony of the Black Scholes model one may keep the property of independent, stationary increments. Notice that in equation (1.1), the exponent is in fact a Lévy process, out of all analytical tractable models which we considered, Lévy process offers the ability to model skewness and kurtosis, hence a candidate that could be considered in replacing Brownian motion is Lévy process. So, we assume

$$S(t) = S(0)e^{L(t)} \tag{1.3}$$

where $L(t)$ is a Lévy process, $L(0) = 0$.

1.1.2 A flexible alternative: Jump-types Lévy processes

There is already a vast literature on Lévy processes. During the last fifteen years, there has been a great revival of research interest in these processes, due to the theoretical development and applications to

[1]The interval is the greatest arbitrage bounds on call option prices

option pricing in Mathematical Finance. Many textbooks, research papers and international conferences were devoted to Lévy processes. Some of the reasons why these processes are important, and should be studied are:

- They can generalise the random walk to continuous time.
- They are the simplest class of processes which paths consist of continuous interspersed with jump discontinuous of random size at random times.
- The structure is rich enough to generalise wider classes of processes, such as semimartingales.
- As prototypes of semimartingales, these are natural models to construct stochastic integrals and to derive stochastic differential equations.
- The structure is very robust in the sense of generalising from the Euclidean space to other spaces, such as Banach and Hilbert spaces, Lie symmetry and quantum groups.

From the perspective of option pricing, a number of models have been developed in adopting more flexible distributions than the Gaussian distribution. Some notable examples on the choices of distributions which have been proposed are the the Normal Inverse Gaussian (NIG) [4], the Meixner process (Schoutens [75]), the four-parameter distribution named CGMY after the names of Carr, Geman, Madan and Yor [12], which was generalized to a six-parameter distribution in [13]. Since most of the proposed processes belong to the family of Lévy processes, a vast literature on option pricing has been developed in replacing the traditional underlying source of randomness, the Brownian motion, by a Lévy process. But an important question in this thesis is the choice of the Lévy process . Some authors suggest to use the Poisson Jump Diffusion Model, first put forward by Merton (1976) [50]. The convenient features of this model is that the diffusion part is responsible for the usual fluctuations in the return series and the jump part accounts for extreme events. However, there are several weaker points of this model. Madan & Seneta (1990) strongly advocates against the use of jump diffusion and stochastic volatility models. They argue that the parameters under diffusion based [2] models are highly unstable due to the inherent infinite variation property [7] and they have gone one step further and have considered pure jumps models with no diffusion component, this is the so-called Variance Gamma (VG) models. These are purely discontinuous models (pure jump) with finite variation and infinite activity (low activity). As the name suggests, these kinds of models have no continuous component, with discontinuities infinite in number. The advocator uses economic analysis, combined with structural mathematical results, to point to the use of pure jump price processes over continuous path processes. In short, under the law of one price, the stock price process is an effective semimartingale. Every semimartingale can be expressed as a time-changed Brownian motion. Hence, the study of price processes is reduced to the study of time-changed Brownian motions. Now, the crux of Madan's argument is that the price trajectory can only continuous if the time change is continuous and locally deterministic. Intuitively, one may view such time changes as measure of economic activity such as the arrival of new information, buy and sell orders or trades. One would expect some randomness and local uncertainty from these activity, pointing to a class of discontinuous price processes. Additionally, if time change is meant to be continuous, then price process would also be continuous, then the process must be Gaussian, locally speaking. Hence, their choice of using a pure jump models is justified. However, in this thesis, we have chosen to work with a Lévy process with finite first moment of the following form

$$L(t) = \theta t + \sigma W(t) + X(t) \tag{1.4}$$

[2]includes Black Scholes, Stochastic volatility and jump-diffusion models

Our rationale for this choice is twofold [24].

First, the price process given by equation (1.3) is required to be a martingale. As it will be illustrated in the rest of the thesis, a constant drift term can be easily computed in satisfying this requirement.

Second, for price process $S(t)$ to be a martingale, both of the expectation of $S(t)$ and the expectation of the exponential of $S(t)$ have to be finite. This is because, Lévy processes we use in finance are required to have finite first moment. We would like a model that can allow the volatility smiles and long-tailness of the return distribution, but not as long-tailness as not to allow for the existence of the first moments. The requirement of finite exponential moment excludes Lévy processes of infinite variation such as the stable processes. As we will illustrate in 2.3, the drift term in the new Lévy-Khintchine formula is simply the expectation of the Lévy process at time 1, i.e. $\mathbb{E}(X(1))$, and $\mathbb{E}(L(t)) = t\mathbb{E}(L(1))$, since the diffusion component and the pure jump integral that is of finite variation are both martingales with expectation of 0.

It shall be stressed that the purpose of this thesis is not to promote the use of alternative processes or models, the aim is to provide enough theory so that jump processes and models built from these results hold no mystery to us, and we can conveniently work with them when needed.

Another assumption of the original work [62] that meant to be relaxed is the assumption of independence between Stock price $S(t)$ and trade volume $U(t)$. This is not necessarily realistic and deserves to be relaxed. Motivated from the modelling of loss and allocated loss adjustment expense (ALAE) amount in the actuarial literature, the original proposal for this thesis in addressing this issue was to use copulas. Copulas are used in probability theory to model dependence between real random variables. It is a useful mathematical tool for modelling the dependence structure of a multivariate distribution separate from the marginal distribution without having to explicitly specify a unified, traditional joint distribution. In the case of VWAP, it may allow us describe the joint distribution of $S(t)$ and $U(t)$ while naturally characterize the dependence structure between the two objects. Since Lévy process is of concern, we considered to use the Lévy process analog form of Clayton copula [82]. However, at a later stage, we decided not to adopt this tool in pricing the VWAP options, but to concentrate purely on the theory of Lévy process for this thesis. The reasons are twofold.

First, based on existing literature on Lévy copula[3], to utilise the notion of Lévy copula, an implicit assumption is that both marginal distributions of $S(t)$ and $U(t)$ need to be identical Lévy processes, which contradict our assumption that $U(t)$ is a squared Ornstein-Uhlenbeck process. In fact, the process of volume is rather complex and can not be reconstructed from market prices [11], and it does not seem to be reasonable to arbitrarily assume volume is of the same Lévy process as stock prices.

Second, although it is well known in actuarial community that copulas plays an important role in modelling dependent risk, the practical implementation in Mathematical Finance and derivative pricing appears to be unsuccessful. The methodology of applying Gaussian copula to credit derivative was known to be one of the reason behind the global financial crisis of 2008-2009.

To incorporate the dependence in some fashion, we subsequently decided to use the classical approach to relate two Wiener processes (See Klebaner (2006) p. 120). The method is quite simple, suppose we know how to generate two independent Gaussian random variables, z_1, z_2, then another two random variables can be generated from z_1, z_2 via the following equations.

$$x_1 = z_1$$
$$x_2 = \rho z_1 + \sqrt{1 - \rho^2} z_2$$

[3]We are only aware of one paper on Lévy copula at the time of this thesis, [82], which is due to Peter Tankov.

Replace the above two random variables with two Brownian motions, we have

$$\begin{aligned} \bar{W}^{(1)}(t) &= \widehat{W}(t) \\ \bar{W}^{(2)}(t) &= \rho\widehat{W}(t) + \sqrt{1-\rho^2}\widetilde{W}(t) \end{aligned} \tag{1.5}$$

It is easy to verify that $\sigma^2(\bar{W}^{(1)}(t)) = \sigma^2(\widehat{W}^{(1)}(t)) = t$, and

$$\mathrm{Cov}(\bar{W}^{(1)}(t), \bar{W}^{(2)}(t)) = \mathrm{Cov}(\bar{W}^{(1)}(t), \rho\widehat{W}(t) + \sqrt{1-\rho^2}\widetilde{W}(t)) = \rho t$$

Remark. The concept of correlated Brownian motion boils down to the use of Pearson's correlation (Linear correlation), which is the most frequently used in practice as a measure of dependence. Nevertheless, we shall stress that the method described above works only when the random variables of interest are Gaussian distributed, unlike copulas, that are invariant under strictly increasing transformations of the underling random variables, the use of linear correlation for dependence modelling is quite misleading in general and shall not be taken as the canonical dependence measure. Fortunately, our setup [4] allows us to utilise Pearson correlation to model the dependence structure between the driving Brownian motions.

In closing, the **difference** between this thesis and the main reference paper [62] is twofold. *First*, we consider a popular alternative to the classical geometric Brownian motion process in governing the dynamics of the stock price, this is the jump-type geometric Lévy process. In particular, we focus on the theory of the Lévy process.
Second, the dependence structure of the stock price and volume is modelled via a linear correlation coefficient.

1.2 Problem Formulation

We consider the so-called VWAP options under a Lévy process, with the expiration date T and the strike level $K > 0$, written on the so-called Volume Weighted Average Price. The problem is to find out the expected payoff at time T of this option:

$$C(T) = (A(T) - K)^+ = \left(\frac{\int_0^T S(t)U(t)dt}{\int_0^T U(t)dt} - K \right)^+$$

where

- $S(t) = S(0)\exp\{rt + L(t)\}$ (Stock price evolves as a Geometric Lévy process)
- $L(t) = mt + \sigma_{\mathrm{BM}}\bar{W}^{(2)}(t) + X^{\mathrm{VG}}(t)$
- $U(t) = X^2(t) + \delta$ (Trade volume evolves as a shifted squared Ornstein Uhlenbeck processes)
- $dX(t) = \lambda(a - X(t))dt + \sigma_{\mathrm{OU}}d\bar{W}^{(1)}(t)$

We assume $m, \lambda,\ \sigma_{\mathrm{BM}},\ \sigma_{\mathrm{OU}},\ a,\ \delta$ are bounded constants and $\delta \geq 0, \lambda > 0$. In particular, λ being the speed of mean reversion, a the long term average of the volume process, $\sigma_{\mathrm{OU}}, \sigma_{\mathrm{BM}}$ are respectively, the the volatility of the volume process and diffusion coefficient of the Brownian motion process. $\bar{W}^{(1)}(t)$,

[4]The two dependent random variables in our setup are two Wiener processes, which are Gaussian distributed with mean 0 and variance 1.

and $\bar{W}^{(2)}(t)$ are standard one-dimensional Brownian motions defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with quadratic variation satisfying $[\bar{W}^{(1)}, \bar{W}^{(2)}](t) = \rho t$.

1.3 Research Objectives and Aims

Although there is a vast literature on pricing of Asian options, as just mentioned, very little literature is found on the pricing of VWAP options. On the other hand, there is already a large and still growing literature on option pricing with Lévy processes in a univariate setting, however, none of these papers deal with the problem where the payoff of the options depends on volume weighted average price value of the underlying asset price. This thesis aims to fill this gap. The objective of this thesis is twofold, first, it has a pedagogical purpose in the sense of exploring basic properties of the Lévy process and constructing the option pricing model from a theoretical (stochastic analysis) perspective. Second, we extend the existing literature on VWAP option pricing to a Lévy process framework. In particular, we utilise the semi-analytical method developed in Novikov et al. (2010) under a Lévy process framework.

The aims of this thesis are:

- To explore the theory of Lévy process and its stochastic calculus applications in Mathematical Finance.
- To derive explicit formulae for the moments of VWAP that describe the distribution of the VWAP.
- To conduct Monte Carlo analysis to verify analytical results on VWAP moments.

1.4 Scopes of Research

- The moment-matching technique
  This is a method to approximate a unknown distribution by a known one. To approximate a Non Gaussian distributed by a lognormal distribution, we choose $\tilde{\mu}$ and $\tilde{\sigma}$ of the lognormal distribution such that the first and second moments match those of the target distribution. If X is a Gaussian distributed random variable with mean 0 and variance 1, then let

  $$A = e^{\tilde{\mu} + \tilde{\sigma} x}$$

  be the lognormal variable approximating our distribution. We proceed to choose $\tilde{\mu}$ and $\tilde{\sigma}$ such that the moments

  $$\mathbb{E}(A^i) = e^{i\tilde{\mu} + i^2 \tilde{\sigma}^2 / 2}$$

  for $i = 1, 2$ yield the first and second moments of the approximated distribution. What remains, is to calculate the moment of the approximated distribution[5]. Nevertheless, we elaborate the moment matching technique in 5.3.

- The simulations and calculations
  Monte Carlo simulation with 10^6 trials are implemented with MATLAB. Numerical integration of multiple integral in exact expression of analytical moments is computed in Mathematica.

[5] In this thesis, the approximated distribution is the VWAP distribution

1.5 Thesis Structure

This thesis is structured as follow: Chapter 2 describes the Lévy processes and its behaviour and stochastic properties. Chapter 3 **details** the literature on VWAP options and provides the background on VWAP in practice. Chapter 4 describes Variance Gamma (VG) model. Chapter 5 describes the pricing setup for the VWAP options under a Geometric Lévy model, the pricing methodology (Moment Matching approach) and derives the analytical formulae for moments of VWAP. Chapter 6 describes the Monte Carlo Simulation of the underlying processes of the VWAP option. Chapter 7 presents numerical results for moments of VWAP and call option prices that have been verified by Monte Carlo simulations. Chapter 8 concludes. The appendix contains three parts.

Appendix A includes

- The necessary mathematics used in this thesis;
- Additional calculations of covariance function.

Appendix B includes

- Mathematica codes for the analytical approximations for the geometric Lévy model (main model) concerned in this thesis.
- MATLAB Monte Carlo simulation codes for the geometric Lévy model (main model) concerned in this thesis.

Appendix C includes

- MATLAB codes in simulating trajectories and option price under the geometric Brownian motion model;
- Mathematica codes for the analytical approximations under the geometric Brownian motion model;
- Mathematica codes for the analytical approximations in response to the change of correlation level.

We also provide a CD that contains all Mathematica Codes, MATLAB codes and the exported data for this thesis.

Chapter 2

Lévy Processes and Geometric Lévy models

2.1 Introduction

Lévy processes are objects from probability theory, which proceed Mathematical Finance, as well as in other fields of science such as Physics (turbulence, laser, cooling), Engineering (telecommunication, queues) and Actuarial Science (insurance risk model). One can think of Lévy processes as continuous time analogues of random walk. The best known Lévy process, the Brownian motion, was introduced in 1900 by Bachelier. Later, in 1959, it was refined by Osborne and applied to stock prices by Samuelson (1965) [73]. In addition to Brownian motion, Mandelbrot (1963) [49] put forward the symmetric stable distribution. Later, pure jump based Lévy processes such as Variance Gamma (Madan and Seneta [48]), normal inverse gaussian (NIG) (Barndorff-Nielsen et al. (1985)[4]) and $CGMY$ [6], were developed and studied. Generalized hyperbolic distributions was introduced with the intention of describing a physical phenomenon: the migration of sand-dunes. Although the Generalised Hyperbolic (GH) distributions were meant for something else, Barndorff-Nielsen inspired Eberlein and Keller to investigate these distributions for modelling stock returns (Eberlein and Keller (1995) [25]). The results are distributions that describe stock returns more accurately than the Brownian motion. The need to advance to these more complicated models primarily comes from the hedging and pricing of options, where it is crucial to be able to determine a practicable risk neutral probability of returns. Upon concluding that stock returns are not Gaussian distributed one is left with the task of exploring models that describe returns in way that allows for accurate hedging and option pricing. Flexibility is the key; What is needed is a model that allows for excess kurtosis and skewness, such models can be found within the Lévy family. There are four main classes of Lévy processes which feature heavily in current mainstream literature on market modelling with Lévy processes [43]. They are jump-diffusion processes (consisting of a Brownian motion with drift plus a independent compound Poisson process), the Generalized Tempered Stable processes (which include Variance Gamma process and CGMY process), Generalized Hyperbolic processes and Meixner processes. In this thesis the focus will be on Variance Gamma (VG) process. There is an extensive literature that describes the theory of Lévy processes with applications to finance, including several excellent reference books. We shall stress that this chapter presents a contribution by aiming to provide an overview of Lévy process and their stochastic calculus applications in Mathematical Finance. Some of the results may not be relevant to this thesis, but useful for applications in general. To serve that purpose, most of the proofs are omitted. For the sections 2.2 to 2.5, we follow Sato (1999) [74], Cont & Tankov (2004) [15] and Papapantoleon (2008) [64]; For the sections 2.6. and 2.7, we follow Cont

& Tankov (2004), Protter (2005) [70] and Øksendal et al (2009) [20].

2.2 Lévy process and Infinite Divisibility

Let us start with the definition of two familiar processes, a Brownian motion and a Poisson process.

A real-valued process $W = \{W(t), t \geq 0\}$ with $W(0) = 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion if the following conditions are satisfied:

- The trajectories of W are $\mathbb{P}$-a.s. continuous.
- It starts at zero: $W(0) = 0$ or $\mathbb{P}(W(0) = 0) = 1$.
- Stationary increments:$\forall \; 0 \leq s \leq t, \; W(t) - W(s) \stackrel{d}{=} W(t+h) - W(s+h)$.
- Independent increment: $\forall \; 0 \leq u \leq s \leq t$, $W(t) - W(s)$ is independent of $\{W(u), u \leq s\}$
- Distribution identity:$\forall \; 0 \leq s \leq t, \; W(t) - W(s) \stackrel{d}{=} W(t-s)$.
- For each $t > 0$, $W(t) \sim N(0, t)$.

A process valued on the non-negative integer $N = \{N(t) : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Poisson process with intensity $\lambda > 0$ if the following are satisfied:

- The trajectories of N are $\mathbb{P}$-a.s. right continuous with left limits (RCLL).
- It starts at zero: $N(0) = 0$ or $\mathbb{P}(N(0) = 0) = 1$.
- Stationary increment:$\forall \; 0 \leq s \leq t, \; N(t) - N(s) \stackrel{d}{=} N(t+h) - N(s+h)$.
- Independent increment: $\forall \; 0 \leq u \leq s \leq t$, $N(t) - N(s)$ is independent of $\{N(u), u \leq s\}$.
- Distribution identity:$\forall \; 0 \leq s \leq t, \; N(t) - N(s) \stackrel{d}{=} N(t-s)$.
- For each $t > 0$, $N(t)$ is Poisson distributed with parameter λt.

The two processes seems to be considerably different from one another. Firstly, Brownian motion has continuous trajectories whereas a Poisson process does not. Secondly, a Poisson process is a non-decreasing process and has trajectories of finite variation[1], whereas a Brownian motion has trajectories of infinite variation.

Nevertheless, they share a lot in common. Both processes have right continuous paths with left limits, start from the origin and both possesses the stationary and independent increment. With the aid of these properties, we are in a position to define a general class of one dimension processes, which are called Lévy processes.

Definition 2.1. A process $X = \{X(t), t \geq 0\}$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if the following properties are satisfied:

- Càdlàg path: The trajectories of X are $\mathbb{P}$-a.s. right continuous with left limits.
- It starts at zero: $X(0) = 0$ or $\mathbb{P}(X(0) = 0) = 1$.
- Stationary increment:$\forall \; 0 \leq s \leq t, \; X(t) - X(s) \stackrel{d}{=} X(t+h) - X(s+h)$.
- Independent increment: $\forall \; 0 \leq u \leq s \leq t$, $X(t) - X(s)$ is independent of $\{X(u), u \leq s\}$.

[1] trajectories of bounded variation over finite time horizons

- Distribution identity:$\forall\ 0 \le s \le t,\ X(t) - X(s) \overset{d}{=} X(t-s)$.

Lévy processes are processes with very strong properties, the first properties can be observed directly from Definition 2.1. That is, every Lévy process has the properties of stationary and independent increments. In the early days of the research literature, Lévy processes were simply referred to as processes with stationary and independent increments. Notice that these properties implies that a Lévy process is a Markov process. With the aid of almost sure right continuity of trajectories, one may show Lévy processes are also strong Markov processes[2].

Based on the definition of Lévy process, one may not be able to see how rich a class of process the class of Lévy process forms. De Finetti[3] [18] introduces the notion of infinite divisible distribution. The property of infinite divisibility is very powerful in the sense that it allows us to construct any Lévy process of interest. A random variable is infinite divisible if it could be written as a sum of n i.i.d. random variables, for all $n \ge 2$. The meaning is that the distribution of $Y_j^{(n)}$ depends only on n, but not on j. Based on this property, the random variables could be reconsidered as a sum of smaller pieces. More precisely,

Definition 2.2 (Infinitely divisible distribution)**.** We say that a real valued random variable Θ has an infinitely divisible distribution if for each $n = 1, 2, ...$, there exists a sequence of i.i.d. random variables $\Theta_1, ..., \Theta_n$ such that

$$\Theta \overset{d}{=} \Theta_{1,n} + ... + \Theta_{n,n}$$

Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law μ of a real valued random variable is infinitely divisible if for each $n = 1, 2, ...$ there exists another law μ_n of a real valued random variable such that $\mu = \mu_n^{*n}$, the n-fold convolution of μ_n.

Theorem 2.1. *All infinite divisible distribution is closed under affine transformations, convolutions and limits.*

The two well known distribution: Poisson and the Gaussian distribution, are infinitely divisible. This can be verified by the respective characteristic functions,

$$\varphi_{\text{gauss}}(u) = \exp\left(i\theta u - \frac{\sigma^2}{2}u^2\right), \tag{2.1}$$

$$\varphi_{\text{pois}}(u) = \exp\left(\lambda(e^{iu} - 1)\right), \tag{2.2}$$

satisfy Definition 2.2.

2.3 Lévy-Khintchine Representation

The full extent to which we may characterize infinite divisible distributions is carried out via their characteristic exponent ψ and this expression is known as the Lévy-Khintchine formula. This formula builds the one-to-one correspondence between the Lévy process and its characteristic function. In this section, we present two versions of the Lévy-Khintchine formulae. First,

Theorem 2.2 (Lévy Khintchine formula 1)**.** *Every infinite divisible distribution μ can be written in the form*

$$\int_{\mathbb{R}} e^{iux}\mu(dx) = e^{-\psi_\mu(u)}, \quad u \in \mathbb{R}$$

[2]The proof is not provided in this thesis as it is very technical and is beyond the scope of this thesis.
[3]Italian probabilist, statistician and actuary.

with

$$\psi_\mu(u) = -i\theta u + \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx)$$

or some $\theta \in \mathbb{R}, \sigma^2 \geq 0$, *and* ν *on* $\mathbb{R}_0$ *such that*

$$\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(dx) < \infty \tag{2.3}$$

The parameter θ, σ^2 *and* ν *uniquely characterize the distribution law* μ. $(\theta, \sigma, \nu(dx))$ *is a generating triplet, where* θ *is the drift coefficient,* σ^2 *the diffusion coefficient and* ν *the Lévy measure.* $\nu(dx)$ *will often be of the form* $k(x)dx$, *and* k *is often referred as the Lévy density.*

From the definition of Lévy process, we see that $X(t)$ is an infinitely divisible random variable. This follows from the fact that for any $n = 1, 2, ...$

$$X(t) = X(t/n) + (X(2t/n) - X(t/n)) + ... + (X(t) - X((n-1)t/n)) \tag{2.4}$$

together with the fact that X has stationary independent increments. Suppose now that we define for all $\theta \in \mathbb{R}, t \geq 0$,

$$\psi_t(u) = -\log \mathbb{E}(e^{iuX(t)})$$

When $t = 1$, we have

$$\psi(u) = -\log \mathbb{E}(e^{iuX(1)})$$

Using equation (2.4) twice we have for any integer m, n that

$$m\psi_1(u) = \psi_m(u) = n\psi_{m/n}(u) \quad \forall \quad m, n \in \mathbb{N}$$

and hence for any rational $t > 0$,

$$\psi_t(u) = t\psi_1(u) \quad \forall \quad t \in \mathbb{Q}$$

where $\psi(u) := \psi_1(u)$ is the characteristic exponent of X_1 which has infinite divisible distribution.

Theorem 2.3. $X(t)$ *is infinitely divisible for a Lévy process* $\{X(t), t \geq 0\}$. *Furthermore*

$$\varphi_{X(t)}(u) = e^{tX_1(u)} = e^{\psi(u)}$$

and

$$\mathbb{E}(X(t)) = t\mathbb{E}(X(1))$$

We see that each Lévy process can be associated with an infinitely divisible distribution. An important issue is that, given an infinitely divisible distribution, can we construct a Lévy process X, such that $X(1)$ has that distribution. The following theorem will answer this question.

Theorem 2.4 (Lévy-Khintchine formula for Lévy process)**.** *Suppose that* $\theta \in \mathbb{R}, \sigma \geq 0$ *and* ν *is a measure concentrated on* $\mathbb{R}_0$ *such that* $\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(dx) < \infty$. *From this triplet define for each* $\theta \in \mathbb{R}$,

$$\psi(u) = -i\theta u + \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx)$$

Then there exists a probability space on which a Lévy process is defined having characteristic exponent φ.

The Lévy process is usually specified by a generating triplet: (θ, σ^2, ν). Notice that if $\sigma^2 > 0$ there is a continuous Gaussian component in the Lévy process X, otherwise, X is a pure jump Lévy process. Pure jump Lévy process can be classified according to the behaviour of Lévy measure.

Proposition 2.5. *Let X be a Lévy process with a generating triplet (θ, σ^2, ν), the there are three cases that one may consider.*

- *Case 1: ν is a finite measure, i.e., $\int_{\mathbb{R}} \nu(dx) < \infty$;*
- *Case 2: ν is not finite, but $\int_{|x|\leq 1} |x|\nu(dx) < \infty$;*
- *Case 3: ν is not finite, and $\int_{|x|\leq 1} |x|\nu(dx) = \infty$.*

Proposition 2.6.

1. *If $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, then almost all paths of X have finite variation.*
2. *If $\sigma \neq 0$ and $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, then almost all paths of X have infinite variation.*

For case 1, if $\sigma = 0$, then the stochastic processes so-called *jump diffusions* are obtained; Otherwise, compound Poisson processes are obtained.

For case 2, if $\sigma \neq 0$, then the process is an infinite activity process of bounded variation. Such processes include the Gamma process, the Variance Gamma process.

Processes fall into case 3 in Proposition 2.5 are *stable processes*, they are not even of bounded variation.

An important result mentioned previously, and will be used in the rest of the thesis is the following:

Theorem 2.7. *If $X = \{X(t), 0 \leq t \leq T\}$ is a Lévy process then for any $t > 0$ and u*

$$\varphi_{X_t} = \mathbb{E}(e^{iuX(t)}) = e^{t\psi(u)} \tag{2.5}$$

$$= \exp\left\{t(i\theta u - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx))\right\} \tag{2.6}$$

where

$$\psi(u) = i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx) \tag{2.7}$$

is the characteristic exponent of X_1, a random variable with an infinite divisible distribution for some $\theta \in \mathbb{R}, \sigma^2 \geq 0, \nu$ on $\mathbb{R}_0$ such that $\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(dx) < \infty$

Notice that the Lévy measure ν is not necessarily a probability measure and so it can either be finite or infinite, hence $\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(dx) < \infty$ is required to ensure the tail of ν are finite. On the other hand, should ν be a infinite measure due to unbounded mass around the origin, then it must at least integrate locally against x^2 for small value of x. The Lévy measure has the interpretation that $\nu(A)$ for any subset $A \in \mathbb{R}$ is the rate which the process takes jumps of size $x \in A$. Alternatively, one can view it as the number of jumps of size falling in A per unit of time. The trajectories are continuous if and only if $\nu = 0$.

The dynamics of the Lévy-Khintchine formula boils down to the integral over the Lévy measure. We see that in equation (2.7), the compensation of small jumps is highlighted in red. Such compensation is necessary in general. Now take $t = 1$, then the Lévy-Khintchine formula becomes

$$\mathbb{E}(e^{iuX_1}) = \exp\left\{i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx)\right\}$$

If the following integrability condition is satisfied,

$$\int_{|x|\leq 1} |x|\nu(dx) < \infty$$

then we no longer need the compensation of small jumps, and the red term can be taken out from the integral and to be added to the drift term, i,e.

$$\mathbb{E}(e^{iuX_1}) = \exp\left\{i\theta' u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)\right\} \tag{2.8}$$

where $\theta' = \theta - iu\int_{\{|x|<1\}} x\nu(dx)$. If a stronger integrability condition holds, i.e.

$$\int_{|x|>1} |x|\nu(dx) < \infty$$

Then we can also compensate large jumps, i.e.

$$\begin{aligned}
\mathbb{E}(e^{iuX_1}) &= \exp\left\{i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx)\right\} \\
&= \exp\left\{i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx) - iu\int_{\mathbb{R}} x\mathbb{I}_{\{|x|<1\}}\nu(dx)\right\} \\
&= \exp\left\{i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx) - iu\int_{\mathbb{R}} x\mathbb{I}_{\{|x|<1\}}\nu(dx) + iu\int_{\mathbb{R}} x\mathbb{I}_{\{|x|>1\}}\nu(dx) - iu\int_{\mathbb{R}} x\mathbb{I}_{\{|x|>1\}}\nu(dx)\right\} \\
&= \exp\left\{i(\theta + \int_{\mathbb{R}} x\mathbb{I}_{\{|x|>1\}})u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx)\right\} \\
&= \exp\left\{i\theta'' u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx)\right\}
\end{aligned}$$

where $\theta'' = \theta + \int_{|x|>1} x\nu(dx)$.

To summarise the above, there are three central concepts of Lévy process that one shall bear in mind before working with Lévy process.

First, the property of *infinitely divisibility* and the *Lévy-Khintchine theorem*. Some may ask why are they necessary and deserve us to study? One main reason is that they are the restrictions one shall be aware of before modelling and inferencing with Lévy process. If we take an infinite divisible distribution, take its parameters as (θ, σ^2, ν), and in that framework we can construct a Lévy process in the form of

$$X(t) = X^{(1)}(t) + X^{(2)}(t) + X^{(3)}(t)$$

where $X^{(1)}$ is a linear trasnformation of a Wiener process with drift, $X^{(2)}$ is a compound Poisson process with jump size at least larger than 1, $X^{(3)}$ is a pure-jump martingale with jump size less than one.

However, if we take any other non infinitely divisible distributions, such as the uniform distribution. then we cannot construct a Lévy process with that distribution. We cannot set up any part of the above formula because θ, σ^2, ν are not present. Hence, it fails at the very beginning of having an incorrect distribution.

Proposition 2.8. *Let X be a Lévy process with a generating triplet (θ, σ^2, ν).Then*

- *X_t has finite p^{th} moment for $p \in (0, \infty)$, i.e. $\mathbb{E}(|X(t)|^p)$, if and only if*

$$\int_{|x|\geq 1} |x|^p \nu(dx) < \infty$$

- *X_t has finite p^{th} exponential moment for $p \in (0,\infty)$, i.e. $\mathbb{E}(e^{pX(t)})$, if and only if*

$$\int_{|x|\geq 1} e^{px}\nu(dx) < \infty$$

Proposition 2.9. *If $\mathbb{E}(|X(t)|) < \infty$, i.e., $\int_{|x|\geq 1}|x|\nu(dx) < \infty$, then $\mathbb{E}(X(t)) = t(\theta + \int_{|x|\geq 1}|x|\nu(dx))$, the process $X(t) - \mathbb{E}(X(t))$ is a martingale.*

Proposition 2.8 indicates that the existence of the moments is determined by the frequency of the big jumps. Proposition 2.9 allows us to compensate big jumps to form a martingale. Hence the Lévy-Khintchine formula takes the form

$$\mathbb{E}(e^{iuX(t)}) = \exp\left\{t(iu\theta'' - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}}(e^{iux} - 1 - iux)\nu(dx))\right\} \tag{2.9}$$

where $\theta'' = \theta + \int_{|x|>1} xd\nu(dx)$.

As mentioned in last chapter, the class of Lévy processes we consider in Mathematical Finance is required to have finite first moment, proposition 2.8 helps us to identify such processes.

Another important concept is the *Lévy measure*. The studies of Lévy measure is important as it allows one to determine whether a process is of finite or infinite variation. As discussed previously, the integrability of Lévy measure carries information about the finiteness of the moments of a Lévy process. The finiteness of the moments of a Lévy process is related to the restriction of the Lévy process to big jumps, i.e. jump size larger than 1 in absolute value. This is a particular useful piece of information in Mathematical Finance as it relates to the existence of a martingale measure.

In the context of financial modelling, we would like to work with the subclass of Lévy process has at least the first moment exist. The rationale is that, for this subclass, a stronger integrability criterion holds for the integral over the Lévy measure, i.e. $\int_{|x|\geq 1}|x|\nu(dx) < \infty$, hence one can compensate the big jumps into the drift term and the Lévy Khintchine formula has a reasonable form where the mean of the process appears as the drift term, i.e.

$$\theta'' = \theta + \int_{|x|>1} xd\nu(dx) = \mathbb{E}(X(1))$$

The new drift term $\theta'' = \mathbb{E}(X(1))$ since the diffusion component and pure jump component in equation (2.9) are martingales.

We will call Lévy process of this type a special semimartingale[4] that can be decomposed into

$$X(t) = \theta t + \sigma W(t) + J(t)$$

where W is a Wiener process and J is a purely discontinuous martingale that is independent of W. The above equation is our choice among all other subclass of Lévy process as it ensures the existence of first moment.

Now we introduce the concept of subordination, following with some examples of Lévy process used in various applications.

2.4 Subordination of Lévy process

The concept of subordination is important as it allows us to transform from one Lévy process to another by time-changing with an increasing Lévy process. More precisely,

[4]As it will be defined in equation (2.15)

Definition 2.3. A real-valued Lévy process is said to be a *subordinator* if it has a.s. nondecreasing trajectories. The generating triplet must satisfy

$$\nu(-\infty,0)=0,\ \ \sigma^2=0,\ \ \int_0^1 x\nu(dx)<\infty,\ \ \bar{\theta}=\theta+\int_0^1 x\nu(dx)>0$$

and the Lévy-Khintchine formula takes the form

$$\mathbb{E}(e^{iuX(t)})=\exp\left\{t(iu\tilde{\theta}+\int_0^\infty (e^{iux}-1)\nu(dx))\right\}$$

Proposition 2.10. *Let τ be a subordinator with Lévy measure ν, drift θ and one-dimensional law f. Let Y be a Lévy process on $\mathbb{R}$ with triplet (θ,σ^2,ν) and one dimensional law g. If τ and Y are independent, define the process*

$$X(t)=Y(\tau(t))$$

Then X is a Lévy process on $\mathbb{R}$ with a generating triplet $(\tilde{\theta},\tilde{\sigma}^2,\tilde{\nu})$.

Many popular Lévy processes are created through subordination. For instance, the Variance Gamma (VG) process is a Wiener process subordinated by a Gamma process [48]. The idea of constructing a VG process through subordination is the following:

Suppose we start with a Wiener process with drift coefficient θ and diffusion coefficient σ

$$B(t;\theta,\sigma)=\theta t+\sigma W(t)$$

Now let's randomise the time index t by letting it follow a gamma process with unit mean rate and variance rate ν, i.e. $\tau_\gamma \sim \Gamma(\frac{t}{\nu},\nu)$. The Gamma process is infinitely divisible, this results a pure jump Lévy process that has an infinite number of jumps in any interval of time:

$$B_{\tau_\gamma}^{(\theta,\sigma)}=X_{\text{VG}}(t)=\theta\tau_\gamma(t)+\sigma W(\tau_\gamma(t))$$

Similarly, the Normal Inverse Gaussian process is a Wiener process subordinated by Inverse Gaussian process [4]. In this case, the time change is given by a Inverse Gaussian distribution. The Inverse Gaussian distribution is infinitely divisible and the resulting pure jump Lévy process is

$$B_{\tau_{\text{IG}}}^{(\theta,\sigma)}=X_{\text{NIG}}(t)=\theta\tau_{IG}(t)+\sigma W(\tau_{IG}(t))$$

The Generalised Hyperbolic process is a Wiener process subordinated by a Generalise Inverse Gaussian process [25]. In this case, the time change is given by a Generalised Inverse Gaussian distribution. The Generalised Inverse Gaussian distribution is infinitely divisible and the resulting pure jump Lévy process is

$$B_{\tau_{\text{GIG}}}^{(\theta,\sigma)}=X_{\text{GH}}(t)=\beta\tau_{GIG}(t)+\sigma W(\tau_{GIG}(t))$$

2.5 Examples of Lévy process

In this section we give some examples of Lévy processes used in finance, economics, risk management and actuarial science.

2.5.1 Gaussian Process

The definition of Gaussian Process can be found in most of the elementary stochastic process textbook (See [8; 39]). Let $X(t) \sim N(\mu t, \sigma^2 t)$, the characteristic exponent of a Gaussian process is given by

$$\psi(u) = i\theta u - \frac{u^2\sigma^2}{2} \quad u \in \mathbb{R}$$

Clearly this characteristic is of the form (2.7) with $\nu(A) = 0, \quad \forall A \in \mathcal{B}(\mathbb{R}_0)$

Gaussian process is infinitely divisible and the following holds:

$$\begin{aligned}
\varphi_{X_t}(u) &= \exp\left(i\theta ut - \frac{u^2\sigma^2 t}{2}\right) \\
&= \exp\left(n(iu\frac{\mu t}{n} - \frac{1}{2}u^2\frac{\sigma^2 t}{n})\right) \\
&= \left(\exp\left(iu\frac{\mu t}{n} - \frac{1}{2}u^2\frac{\sigma^2 t}{n}\right)\right)^n \\
&= \left(\varphi_{X_{(t/n)}}(u)\right)^n
\end{aligned}$$

where $X^{(t/n)} \sim N(\frac{\mu t}{n}, \frac{\sigma^2 t}{n})$.

2.5.2 Poisson Process

Definition 2.4 (Poisson process)**.** The counting process N_t with $N_0 = 0$ is said to be a Poisson process having intensity rate $\lambda > 0$ if N_t has independent increments and for all $s, t \geq 0$

$$P(N_t - N_s = k) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!} \tag{2.10}$$

for $k \in \{0, 1, ...\}, t \in [0, \infty)$ and $s \in [0, t]$

Consider the Dirac measure $\delta_c : \mathcal{B}(\mathbb{R}) \longmapsto [0, \infty)$ where

$$\delta_c(A) = \mathbb{I}_{\{A\}}(c) = \mathbb{I}_{c \in A}$$

The characteristic exponent of a Poisson distribution is

$$\psi(u) = \lambda(e^{iu} - 1) \quad u \in \mathbb{R}$$

is of the form (2.7) with $\theta = \sigma^2 = 0$ and $\nu(A) = \lambda\delta_1(A)$

Poisson process is infinitely divisible and the following holds:

$$\begin{aligned}
\varphi_{X_t}(u) &= \exp\left(\lambda t(e^{iu} - 1)\right) \\
&= \left(\exp(\frac{\lambda t}{n}(e^{iu} - 1))\right)^n \\
&= \left(\varphi_{X_{(t/n)}}(u)\right)^n
\end{aligned}$$

where $X^{(t/n)} \sim \text{Poisson}(\frac{\lambda t}{n})$

2.5.3 Compound Poisson Process

Definition 2.5 (Compound Poisson Processes)**.** A stochastic process X_t is said to be a compound Poisson process if it can be represented as

$$Y_t = \sum_{k=1}^{N_t} \xi_k \tag{2.11}$$

for $t \in [0, \infty)$. A Compound Poisson Processes generates a sequence of pairs $(\tau_k, \xi_k)_{k\in\mathbb{N}}$ of *jump times* τ_k and *marks*[5] ξ_k. N_t is a Poisson process with intensity rate λ and the *marks* ξ_k are iid r.v.'s which are also independent of N_t.

The characteristic exponent of a compound Poisson distribution is given by

$$\psi(u) = \int_{\mathbb{R}} (e^{iux} - 1)\lambda f(dx)$$

where f is the law of the jumps. This characteristic exponent is also of the form (2.7) with $\theta = \int_{-1}^{1} x\lambda f(dx), \sigma^2 = 0$ and $\nu(dx) = \lambda f(dx)$.

Compound Poisson process is infinitely divisible. To show this, first let the characteristic function of marks be $\mathbb{E}(e^{iux}) = \int_{\mathbb{R}} e^{iux} f(dx)$, then conditioning on there being n jumps on the interval $[0, t]$, we have

$$\mathbb{E}(e^{iuX_t} | N_t = n) = \mathbb{E}(e^{iu(X_1 + \ldots X_n)} | N_t = n) = \left(\int_{\mathbb{R}} e^{iux} f(dx)\right)^n,$$

by the independence of the marks $x_1, ..., x_n$.
Recall that, for a Poisson process, $\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$. By the tower law (See Appendix A: Mathematical Preliminaries), the unconditional expectation is then

$$\begin{aligned}
\mathbb{E}(e^{iuX_t}) &= \mathbb{E}(\mathbb{E}(e^{iuX_t} | N_t)) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N_t = n) \mathbb{E}(e^{iuX_t} | N_t = n) \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \left(\int_{\mathbb{R}} e^{iux} d(dx)\right)^n \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t (\int_{\mathbb{R}} e^{iux} f(dx)))^n}{n!} \\
&= \exp\left\{-\lambda t + \lambda t \left(\int_{\mathbb{R}} e^{iux} f(dx)\right)\right\} \\
&= \exp\left(\lambda t \int_{\mathbb{R}} (e^{iux} - 1) f(dx)\right) \\
&= \exp\left(\frac{\lambda t}{n} \int_{\mathbb{R}} (e^{iux} - 1) f(dx)\right)^n \\
&= \left(\varphi_{X_{(t/n)}}(u)\right)^n
\end{aligned}$$

2.5.4 Generalised Inverse Gaussian Distribution

This family is a generalisation of the Inverse Gaussian (IG) distribution and has been studied extensively in Jørgensen (1982) [37]. The Generalised Inverse Gaussian Distribution is a three parameter distribution, it arises in the context of the first passage time of a diffusion process, when the drift and variance of displacement per unit time are dependent upon the current position of the particle [1]. It is proven to be infinitely divisible [5] and generate a Lévy process (a subordinator). The three parameter

[5] *'marks'* stands for the size of the jumps, see [67]

GIG distribution has the density function as follows:

$$f(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-(ax+b/x)/2}, \quad x > 0$$

where $K_p(\cdot)$ is a modified Bessel function of the second kind with index p. The variation of the parameter of the GIG distribution is [26]

$$\theta_p = \begin{cases} \{(b,a) \mid b \geq 0, a > 0\}, & \text{if } p > 0 \\ \{(b,a) \mid b > 0, a > 0\}, & \text{if } p = 0 \\ \{(b,a) \mid b \geq 0, a \geq 0\}, & \text{if } p < 0 \end{cases}$$

The above means that if $p > 0, b = 0$, the GIG distribution reduces to a Gamma distribution; If $a = 0, p < 0$, the GIG distribution reduces to a Reciprocal Gamma distribution. When $p = -\frac{1}{2}$, we obtain the Normal Inverse Gaussian distribution. Other important cases are $p = 0$ (the hyperbola distribution) and $p = 1$.

The GIG distribution has a characteristic exponent of the form (2.6) with Lévy measure

$$\nu(dx) = \frac{1}{x}(b \int_0^\infty e^{-xt} g_p(2bt)dt + (0,p)^+)e^{-ax/2}dx$$

where

$$g_p(y) = \frac{1}{\frac{\pi^2}{2} y (J_{|p|}^2(\sqrt{y}) + M_{|p|}^2)(\sqrt{y})}, \quad y \geq 0$$

where J and M are modified Bessel functions. We refer to Abramowitz and Stegun (1970) for further discussion on Bessel functions.

Note that we may not be able to determine whether the process is of finite variation or infinite variation as the Lévy measure for this class involved special functions and it is quite involved.

2.5.5 Generalised Hyperbolic Distribution

The generalised hyperbolic distribution (GH) is a continuous probability distribution defined as the normal variance mean mixture where the mixing distribution is the generalise inverse Gaussian distribution, i.e.

$$Y = \alpha + \beta V + \sigma\sqrt{V}X$$

where α, $\beta \in \mathbb{R}$, $\sigma > 0$ and random variable X and V are independent, $X \sim N(0,1)$. V is a continuous distribution. In this case V is a Generalised Inverse Gaussian (GIG) distribution having the density of the form equation (2.5.4). The conditional distribution of Y given V has mean $\alpha + \beta V$ and variance V. Moments of any order exist.

The GH process can be constructed via subordination by setting the time change as a GIG distribution. If we define

$$X_{\text{GH}}(t) = \mu t + \beta\tau_{\text{GIG}}(t) + W(\tau_{\text{GIG}}(t))$$

where $W(t)$ is a Wiener process and $\tau_{\text{GIG}}(t)$ is generated by GIG process.

This family was first introduced by Barndorff-Nielsen (1977) [5]. In term of the α, β, δ parameterisation, the density function of the Generalised Hyperbolic (GH) Distribution is given in terms of modified

Bessel function of the second kind, denoted by K_λ

$$p(x;\lambda,\alpha,\beta,\delta,\mu) = \frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi}K_\lambda(\delta\gamma)} e^{\beta(x-\mu)} \times \frac{K_{\lambda-1/2}(\alpha\sqrt{\delta^2+(x-\mu)^2})}{(\sqrt{\delta^2+(x-\mu)^2}/\alpha)^{1/2-\lambda}}$$

where the parameter $\alpha > 0$ determines the shape, β with $0 \le |\beta| < \alpha$ the skewness and $\mu \in \mathbb{R}$ the location. $\delta > 0$ is the scaling parameter and $\gamma = \sqrt{\alpha^2 - \beta^2}$. As the name suggests the GH distribution is of a very general form, it embraces many subclasses, respectively the student's t-distribution, the Laplace distribution, the hyperbolic distribution, the normal-inverse Gaussian distribution and the VG distribution. The main applications of GH distribution are those when require sufficient probability of far-field behaviour, which it can model due to its semi-heavy tails. All of the mentioned subclasses of the GH distribution have been widely used in modelling financial returns and risk processes, due to its semi-heavy tails. The parameter $\lambda \in \mathbb{R}$ defines the subclasses,as it controls the heaviness of the tails. For $\lambda = 1$ we have the hyperbolic distribution, for $\lambda = -1/2$ we have the normal inverse Gaussian. For $\delta = 0$ we have the VG distribution. It is worth noticing that the two subclasses of the GH distribution Normal inverse Gaussian and VG are closed under convolution.
The name of this family comes from the fact that, for $\lambda = 1$, the logarithm of the density gives an hyperbola, unlike the case of a Gaussian distribution where gives a parabola. As a consequence, the tail of the distribution decays slowly with respect to the Normal. By changing the axis of the hyperbola we get positively and negatively skewed densities.
The characteristic exponent is of the form (2.7) and is given by

$$\psi(u) = iu\mathbb{E}(GH) + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux)\nu(dx)$$

where $\mathbb{E}(GH) = \mu + \frac{\delta\beta K_{\lambda+1}(\delta\gamma)}{\gamma K_\lambda(\delta\gamma)}$. The Lévy measure ν in terms of Bessel functions of the first and second kind is given as follows:

$$\nu(dx) = \begin{cases} \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{e^{-|x|\sqrt{2y+\alpha^2}}}{\pi^2 y(J_\lambda^2(\delta\sqrt{2y})+M_\lambda^2(\delta\sqrt{2y}))}dy + \lambda e^{-\alpha|x|}\right)dx, & \text{if } \lambda \ge 0 \\ \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{e^{-|x|\sqrt{2y+\alpha^2}}}{\pi^2 y(J_{-\lambda}^2(\delta\sqrt{2y})+M_{-\lambda}^2(\delta\sqrt{2y}))}dy\right)dx, & \text{if} \lambda < 0 \end{cases}$$

J and M are modified Bessel functions. We refer to Abramowitz and Stegun (1970) for further discussion on Bessel functions.

Again, similar to the case of GIG, the complicated expression of the Lévy measure does not allow us to determine whether the process is of finite variation or infinite variation.

Note that the Gaussian coefficient σ^2 is 0. Another remark of this family is that a random variable X gaving a generalised hyperbolic distribution can be written as a mean-variance mixture of a normal distribution. That is, X is conditionally distributed as a normal $N(\mu + \beta\sigma^2, \sigma^2)$ where, in turn, σ^2 has a generalised inverse Gaussian distribution.

2.5.6 Variance Gamma process

We end this section with a example of the most important Lévy process, being the VG process. The VG process is first introduced by Madan and Seneta (1990). The characteristic exponent of a VG distribution

$$\psi(u) = -\frac{1}{\nu}\ln\left(1 + \frac{u^2\sigma^2}{2}\nu - iu\theta\nu\right) = \ln\left(\frac{1}{1 + \frac{u^2\sigma^2}{2}\nu - iu\theta\nu}\right)\Big/\nu, \; u \in \mathbb{R},$$

and the Lévy measure is

$$\nu(dx) = \begin{cases} C_1 \frac{e^{-C_3 x}}{x} dx, & \text{if } x > 0 \\ C_2 \frac{e^{C_4 x}}{x} dx, & \text{if } x < 0 \end{cases}$$

Nevertheless, VG process will be discussed in full detail in Chapter 4.

2.6 Stochastic Calculus for Lévy processes

Given a monetary asset whose price is represented by a stochastic process $S = \{S(t), t \in [0,T]\}$, two problems related to S are often of concern:

- Construction of trading strategies involving S.
- Analysis of synthetic products whose value depends on S

we need to have some tools that allows us to perform transformations of the Lévy processes. As in classical diffusion model, stochastic integrals and the Itô formula play a central role. To describe trading strategies against S, we need stochastic integrals. To describe the dynamics of a derivative instrument whose value depends on $S(t)$, we need Itô Calculus. To perform transformation under Lévy based model, the same principal holds. In this section, Some basic notions of semimartingales (as a generalisation of Lévy process) are introduced. In particular, we focus on the construction of stochastic integral with respect to a semimartingale. Following, Itô formulae for Lévy process are presented.

2.6.1 Semimartingales

All Lévy processes are semimartingales because any Lévy process can be splitted into a sum of squared integrable martingale and a finite variation process. Semimartingales is a very rich class of processes. The class of semimartingales is considerably stable under stochastic integration and nonlinear transformation, it is also stable under other operations such as change of measure and time change. Hence it turns out to be sufficient to work with semimartingales to model finite dimensional problems that usually appear in finance, risk management and actuarial science [67]. In short, a semimartingale is simply a local martingale plus a process of finite variation. More precisely,

Definition 2.6 (Semimartingales)**.** A regular right-continuous with left limits (càdlàg) adapted process is a semimartingale if it can be represented as a sum of two processes: a local martingale M_t and a process of finite variation A_t, with $M(0) = A(0)$, and

$$S(t) = S(0) + M(t) + A(t) \tag{2.12}$$

for all $t \in [0, \infty)$

In addition, a semimartingale of the form (2.12) can be decomposed into continuous and discontinuous parts [67]. This means that if discontinuous part of $A(t)$ and $M(t)$ are represented as

$$A_t^d = \sum_{0 \le s \le t} \Delta A_s \quad \text{and} \quad M_t^d = \sum_{0 \le s \le t} \Delta M_s$$

then processes A and M can be split into a continuous and discontinuous part:

$$A_t = A_t^c + A_t^d \tag{2.13}$$

$$M_t = M_t^c + M_t^d \tag{2.14}$$

Definition 2.7 (Special semimartingales)**.** A semimartingale S is said to be a special semimartingale if it can be written

$$S(t) = S(0) + M(t) + A(t) \tag{2.15}$$

where $M(0) = A(0) = 0$. $M(t)$ is a local martingale, and $A(t)$ is *predictable* and of finite variation.

The class of semimartingales includes a wide range of stochastic processes such as Markov chain, diffusion processes and Lévy processes. By equation (2.12) and Lévy Ito decomposition we see that a Lévy process is clearly a semimartingale.
It should be stressed that, for a semimartingale S, the process of jumps is defined as $\Delta S(t) = S(t) - S(t-)$. As far as trading strategies are concerned, it is convenient to work with a stochastic integral with respect to a semimartingale. Suppose we hold a portfolio of m asset: $S(t) = (S_t^1, ..., S_t^m)$. Each $S_t^1, ..., S_t^m$ are càdlàg. This investment portfolio lasts for one year, i.e. $T = 1$, the intervals $[0, t]$ is discretized time intervals $T_0 = 0, T_1, T_2, ..., T_n = T$. Let ζ^j be any strategy in the j-th stock, $j = 1, 2, ..., m$. The trade portfolio being considered is a m dimensional vector describing the amount of each share held by us. In other words, we hold m stocks and the value of the stocks is described by the vector $S(t) = (S_t^1, ..., S_t^m)$; The positions taken can be described by vector $\zeta(t) = (\zeta_t^1, ..., \zeta_t^m)$. So at any time $t \in [T_0, T_n]$,the value of of each j-th position in our portfolio is given by $\zeta^j S_t^j$. By summing up over j, the value of the entire portfolio at any time is

$$V_t(\zeta) = \sum_{j=1}^{m} \zeta^j S_t^j = (\zeta \cdot S)(t) := \int_0^t \zeta(u) dS(u)$$

Now let's reverse the previous portfolio by shorting m shares. The second portfolio $\pi_t(\zeta)$ is constructed simultaneously to the investment portfolio $V_t(\zeta)$ at time 0, assume each share in the $\pi_t(\zeta)$ has a value of 0 at time 0, and the price is fixed over the rest of the holding period up to $T = 1$, in other words, the turnover between each two transaction T_k and T_{k+1} is constant and hence the portfolio value remains unchanged. Mathematically, the amount of each asset held at date t could be described by a simple predictable process $\zeta(t)$ and expressed as

$$\zeta(t) = \zeta(0)\mathbb{I}_0 + \sum_{k=0}^{n-1} \zeta_k \mathbb{I}_{(T_k, T_{k+1}]}$$

We see that the indicator function is of the form $\mathbb{I}_{(T_k, T_{k+1}]}$ (left continuous) as opposed to $\mathbb{I}_{[T_k, T_{k+1})}$ (right continuous). This operation allows the value at t_k to be defined before the jump $\zeta(T_k) := \zeta(T_{k-})$, which is reasonable. In practice, the transaction dates can be viewed as stopping times (non-anticipated random times) in the sense that they form a random partition over $[0, T]$. Since the new position ζ_k is chosen based on the information available up to T_k, hence ζ_k is $\mathcal{F}_{T_k}$ measurable. From a practitioner's point of view, regardless the fact that the transaction settles at $t = T_k$, the portfolio is still described by ζ_{k-1}. It takes new value ζ_k right after settlement. Hence, the indicator function is càglàd rather than càdlàg, which is practically sensible. It shall be stressed that stochastic processes of the above equation form are called *simple predictable processes*. A predictable process is important in constructing stochastic integrals because it is the only type of process that can be integrated w.r.t. a semimartingale. The offsetting portfolio $\pi_t(\zeta)$ is constructed in reversing the first portfolio, so again let ζ^j be the strategy in the j-th share. At time 0 the value of the strategy has value 0 (or any other initial value). The position for j-th stock at any transaction date over the trading horizon can be described as follows.

$t = T_0 = 0$ $\quad$ ζ_0^j position in share with price S_0^j $\quad$ $\zeta_0^j S_0^j$

$$t = T_1 \qquad \zeta^j_{T_1} \text{ position in share with price } S^j_0 \qquad \zeta^j_{T_1} S^j_{T_1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$t = T_n \qquad \zeta^j_{T_n} \text{ position in share with price } S^j_0 \qquad \zeta^j_{T_n} S^j_{T_n}$$

Between T_k and T_{k+1}, the position of stock in the portfolio is ζ_k and the asset moves by $(S(T_{k+1}) - S(T_k))$ so the the profit gained from selling each k can be conveniently expressed by $\zeta_k \cdot (S(T_{k+1}) - S(T_k))$. For instance, the value of our ζ^j_0 position at time T_1 equals

$$\pi_{T_1} = \zeta^j_0 (S(T_1) - S(T_0))$$

After n steps, the investor starting with a position ζ_0 following the strategy ζ will have accumulated capital at the end of the trading gorizon T equals to :

$$\pi_T = \sum_{k=0}^{n-1} \zeta^j_{T_k} (S(T_{k+1}) - S(T_k)) \underset{n\to\infty}{\to} \int_0^T \zeta^j_n dS \tag{2.16}$$

The gain process associated with strategy ζ is denoted as $\pi_t(\zeta)$. For the stochastic integral to be interpreted as the gain process of the strategy ζ, the portfolio ζ_k should be constructed at the beginning of the period, T_k.[6] The gain process is described by equation (2.16) is the one dimensional *stochastic integral* of the predictable process ζ w.r.t. S and denoted by

$$\pi_t(\zeta) = \sum_{k=0}^{n-1} \zeta_k (S(t_{k+1}) - S(t_k)) \to \int_0^t \zeta dS = (\zeta \cdot S)(t) = V_t(\zeta) \text{ for any } \ t_k < t \le t_{k+1} \tag{2.17}$$

A convenient notation for the stochastic integral in equation (2.17) is $I_S(\zeta)$. Equation (2.17) indicates that the stochastic integral $\int_0^t \zeta dS$ represents the capital accumulated between 0 and t by following strategy ζ. A strategy $(\zeta_t)_{t\in[0,T]}$ can only be self-financing if the cost associated with the strategy, $C_t(\zeta)$ is a.s. 0, in other words, the worth of the portfolio, i.e. $V_t(\zeta) = \zeta_t S(t)$ must equal to $\pi_t(\zeta)$.

The stochastic integral of predictable process ζ with respect to $S : \int_0^t \zeta dS$ represents the capital accumulated between 0 and t by the strategy H. In other words, equation (2.17) represents a gain process associated with strategy H.

The stochastic integral not only represents a gain process associated with a particular trading strategy, but also can be used as a means of constructing new processes and new martingales from the old ones: given a nonanticipating càdlàg process $\{L(t), t \in [0,T]\}$ one can build new processes $\int_0^t \sigma_s dX_s$ by choosing (simple) predictable process $\{\sigma(t), t \in [0,T]\}$. Here $L(t)$ can be viewed as the "source of randomness" and "$\sigma(t)$" can be viewed as the "diffusion coefficient". Starting with a simple stochastic process L such as a Lévy process, this procedure can be used to build stochastic models with desired properties. The following result shows that if the asset price is modelled as a stochastic integral $S(t) = \int_0^t \sigma dL$ with respect to a "source of randomness" then the gain process of any strategies involving S can also be expressed as a stochastic integral with respect to L.

Proposition 2.11 (Associativity). *Let $\{L(t), t \in [0,T]\}$ be a real-valued càdlàg process and $\sigma(t), t \ge 0$ and $\{\zeta(t), t \ge 0\}$ be real-valued simple predictable process. Then $S(t) = \int_0^t \sigma dL$ is a càdàg process and*

$$\int_0^t \zeta_u dS_u = \int_0^t \zeta_u \sigma_u dL_u$$

[6]hence ζ_k can be known at T_k while the variation of share $S(T_{k+1}) - S(T_k)$ is only known at the end of the period, T_{k+1}.

The relation $S_t = \int_0^t \sigma(s)dL(s)$ is often abbreviated to a "diffrential" $dS_t = \sigma_t dL_t$, which should be understood as a shorthand for the integral notation.

So far we have posed no restrictions on the process S in order to make sure that the stochastic integral is well behaved. By well behaved we mean the stability: a small change in process ζ should lead to a small change in the stochastic integral $I_S(\zeta)$. It is known that not all processes satisfy this criteria ([15], p.253). The processes that do satisfy this criteria are so-called semimartingales. More precisely, the strategy ζ is required to converge uniformly, then $I_S(\zeta)$ should converge in probability. A more mathematically formulated definition of semimartingale is the following:

Definition 2.8 (Semimartingale)**.** A nonanticipating càglàd process S is called a semimartingale if the stochastic integral of simple predictable processes w.r.t. S:

$$\zeta(t) = \zeta(0)\mathbb{I}_0 + \sum_{k=0}^{n-1} \zeta_k \mathbb{I}_{(T_k, T_{k+1}]}$$

verifies the following continuity property: for every $\zeta^n, \zeta \in S([0,T])$ if

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |\zeta_t^n(\omega) - \zeta_t(\omega)| \underset{n\to\infty}{\to} 0 \tag{2.18}$$

then

$$\int_0^T \zeta^n dS \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \int_0^T \zeta dS \tag{2.19}$$

If the continuity property above does not hold when modelling asset price by $S(t)$, a very small error in the composition of the strategy will lead to a enormous loss in the portfolio. Hence, semimartingales are often the preferable class of stochastic process in continuous time trading.

As was shown when we described the jump part of Lévy processes, the sample paths of the process may not always be of finite variation (a well-known example could be Brownian motion). Hence we need to consider the quadratic variation of the process to perform a change of variables. Recall the quadratic variation of a stochastic process X_t over interval $[0,t]$ in general is defined as.

$$[X,X](t) = \lim_{\delta_n\to 0} \sum_{k=1}^{n} (X(t_k^n) - X(t_{k-1}^n))^2 \tag{2.20}$$

In the case of a semimartingales (and hence Lévy processes), quadratic variation can be defined in accordance to Protter (2005) p.66 [70].

Definition 2.9 (Quadratic Variation of semimartingales)**.**

1. The quadratic variation process of a semimatingale X is the nonanticipating càglàd process defined by

$$[X,X](t) = |X(t)|^2 - 2\int_0^t X(u-)dX(u) \tag{2.21}$$

2. The quadratic variation of semimartingales X and Y is defined by

$$[X,Y](t) = X(t)Y(t) - X(0)Y(0) - \int_0^t X(s-)dY(s) - \int_0^t Y(s-)dX(s)$$

where $p^n = (t_0^n = 0 < t_1^n < ... < t_{n+1}^n = T)$ is a sequence of partitions of $[0,T]$ such that
$|p^n| = \sup_k |t_k^n - t_{k-1}^n| \to 0$ as $n \to \infty$

Remark 2.1. $X(s-)$ is used in the integrand. This is necessary because $X(s)$ is not a predictable process, whereas $X(s-)$ is. (See Protter (2005) [70])

Notice that the Lévy process could be decomposed into a continuous part and a discontinuous part. This implies that we can decomposed the quadratic variation into a continuous part and a discontinuous part. More precisely,

Definition 2.10. For a semimartingale X, the process $[X,X]^c$ denote the path-by-path continuous part of $[X,X]$. We then have

$$[X,X](t) = [X,X]^c(t) + \sum_{s\in(0,t]} (\Delta X(s))^2$$

If $[X,X]^c(t) = 0$, then X is called a (quadratic) pure jump process. [7]

Corollary 2.12. *If X is a semimartingale, then for each t, $\sum_{s\leq t}(\Delta X(s))^2 < \infty$.*

It shall be stressed that $[X,X](t)$ is a random process, as is shown by the following examples.

Example 2.1 (Quadratic variation of a continuous Lévy process)**.** Let X be a Lévy process with a generating triplet $(\theta,\sigma^2,0)$, i.e. $X(t) = \theta t + \sigma W(t)$, then the quadratic variation is given by $\sigma^2 t$.

Example 2.2 (Quadratic variation of a pure jump Lévy process)**.** Let X be a Lévy process with a generating triplet $(\theta,0,\nu)$, i.e. $X(t) = \theta t + J(t)$, where $J(t) = \sum_{k=1}^{N(t)} \xi_k$ is a compound Poisson process, $N(t)$ is a counting process, ξ_k denotes jump size, $N(t) \perp \xi_k$, then the quadratic variation is given by

$$[X,X](t) = \sum_{k=1}^{N(t)} |\xi_k|^2 = \sum_{s\in[0,t]} |\xi_k|^2 = \sum_{s\in[0,t]} |\Delta X(s)|^2$$

More generally, it is not hard to show that the same formula holds for every finite variation process X:

Remark 2.2. If X is a process of finite variation, then the quadratic variation is given by

$$[X,X](t) = \sum_{s\in[0,t]} |\Delta X(s)|^2$$

Example 2.3 (Quadratic variation of a general Lévy process)**.** Let X be a Lévy process with a generating triplet (θ,σ^2,ν), i.e. $X(t) = \theta t + \sigma W(t) + J(t)$, then the quadratic variation is given by

$$[X,X](t) = \sigma^2 t + \sum_{0<s<t} |\Delta X(s)|^2 = \sigma^2 t + \int_0^t \int_{\mathbb{R}} z^2 N_X(ds,dz)$$

Let's now consider the case when a semimartingale X has jumps. First, define jumps as the difference

$$\Delta X(t) = X(t) - X(t-)$$

Note that at the jump time the value of the Itô integral increases by the value of the integrand before the jump multiplied by the jump size of the integrator. That is

$$\Delta I_{\zeta,X}(t) = I_{\zeta,X}(t) - I_{\zeta,X}(t-) = \zeta(t-)\Delta X(t) \quad \forall \quad t\in[0,\infty)$$

Notice that when describing the jump part of Lévy processes, the sample paths of the process may not always be of finite variation (which is never the case, when considering a process containing a diffusion

[7]See Protter (2005) p.70.

component). Hence, one needs to consider quadratic variation of the process to perform a change of variable. The following defines quadratic variation.
Consider a pure jump process of bounded variation, X. Here X need not be a Lévy process. We can write

$$X(t) = X(0) + \sum_{s\leq t}(X(s) - X(s-)).$$

So, X has no continuous component and the value of the process equals to its initial value plus the sum of its jumps. Now consider, $f(X(t))$ for some real-valued function, f. Then, $f(X(t))$ is also a pure jump process of bounded variation, and

$$f(X(t)) = f(X(0)) + \sum_{s\leq t}(f(X(s)) - f(X(s-)))$$

This is just Itô lemma for the pure jump case:

$$df(X(t)) = f(X(t)) - f(X(t-)).$$

Definition 2.11 (Quadratic covariation)**.** Given two semimatingales X and Y, denoted $[X,Y]$ is the semimartingale defined by

$$[X,Y](t) = X(t)Y(t) - X(0)Y(0) - \int_0^t X(s-)dY(s) - \int_0^t Y(s-)dX(s) \tag{2.22}$$

where $p^n = (t_0^n = 0 < t_1^n < ... < t_{n+1}^n = T)$ is a sequence of partition of $[0,T]$ such that
$|p^n| = sup_k|t_k^n - t_{k-1}^n| \to 0$ as $n \to \infty$

Proposition 2.13 (Itô product rule for Semimartingales)**.** *If X, Y are semimartingales then*

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s-)dY(s) + \int_0^t Y(s-)dX(s) + [X,Y](t) \tag{2.23}$$

Proof. Define quadratic covariation as a limit over an increasingly fine partition of $[0,t]$

$$\begin{aligned}
[X,Y](t) &= \lim_{n\to\infty}\sum_{k=0}^{n-1}(X(t_{k+1}) - X_(t_k))(Y(t_{k+1}) - Y(t_k)) \\
&= \lim_{n\to\infty}\sum(X(t_{k+1})Y(t_{k+1}) - X(t_{k+1})Y(t_k) - X(t_k)Y(t_{k+1}) + X(t_k)Y(t_k)) \\
&= \lim_{n\to\infty}\sum(X(t_{k+1})Y(t_{k+1})) - \sum(X(t_k)(Y(t_{k+1}) - Y(t_k)) - \sum(Y(t_k)(X(t_{k+1}) - X(t_k)) \qquad (2.24) \\
&= X(t)Y(t) - X(0)Y(0) - \int_0^t X(s-)dY(s) - \int_0^t Y(s-)dX(s) \qquad (2.25)
\end{aligned}$$

Note that last two sums in equation (2.24) converge to the Itô integral, that is,

$$\sum(X(t_k)(Y(t_{k+1}) - Y(t_k)) \overset{n\to\infty}{\to} \int_0^t X(s-)dY(s)$$

$$\sum(Y(t_k)(X(t_{k+1}) - X(t_k)) \overset{n\to\infty}{\to} \int_0^t Y(s-)dX_s$$

Hence

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s-)dY(s) + \int_0^t Y(s-)dX(s)$$

□

Example 2.4 (Quadratic covariation of correlated Brownian motions)**.** If $B^{(1)}(t) = \sigma_1 W^{(1)}(t)$ and $B^{(2)}(t) = \sigma_2 W^{(2)}(t)$, where $W^{(1)}, W^{(2)}$ are standard Wiener processes with correlation ρ, then

$$[B^{(1)}, B^{(2)}](t) = \rho\sigma_1\sigma_2 t$$

Definition 2.12 (Itô formula for Semimartingales)**.** Let X be a semimartingale, and $f = f(x,t) \in C^{2,1}(\mathbb{R})$. Then,

$$\begin{aligned} df(X(t),t) = &f_x(t,X(t))dX(t) + f_t(t,X(t))dt \\ &+ \frac{1}{2} f_{xx}(t,X(t))d[X,X]^c(t) \\ &+ f(t,X(t)) - f(t,X(t-)) - f_x(t,X(t-))(X(t) - X(t-)) \end{aligned}$$

Where X^c is the continuous part (or equivalently the continuous martingale part) of X. In integral form, this becomes,

$$\begin{aligned} f(X_t,t) = &f(0,X(0)) + \int_0^t f_x(s,X(s-))dX_s + \int_0^t f_s(s,X(s))ds \\ &+ \frac{1}{2} f_{xx}(s,X(s))d[X,X]^c(s) \\ &+ \sum_{s\le t}(f(s,X(s)) - f(s,X(s-)) - f_x(s,X(s-))(\Delta X(s))) \end{aligned} \tag{2.26}$$

2.6.2 Lévy processes

In this subsection, we briefly introduce the notion of **Jump Measure** [20]. It appears to be quite technical. However, it is useful in the sense that it helps us to perform many computations. Define the jump of Lévy process $L(t)$ as

$$\Delta L(t) := L(t) - L(t-)$$

Let $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the σ-algelbra generated by the family of all Borel subsets $V \subset \mathbb{R}$, such that $\bar{V} \subset \mathbb{R}_0$. If $V \in \mathcal{B}(\mathbb{R}_0)$ and $t > 0$, we define the number of jumps of size $\Delta L(s) \in V$ for any s in $0 \le s \le t$. Since the paths of L

$$N(t,V) := \sum_{0\le s\le t} \mathbb{I}_V(\Delta L(s)) \tag{2.27}$$

that is, the number of jumps of size $\Delta L(s) \in V$ for any s in $0 \le s \le t$. Since the paths of L are càdlàg we see that $N(t,V) < \infty$ for all $V \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{V} \subset \mathbb{R}_0$; see, e.g. [74]. Moreover, equation (2.27) defines a Poisson random measure N on $\mathcal{B}(0,\infty) \times \mathcal{B}(\mathbb{R}_0)$ in a natural way:

$$(a,b] \times V \longmapsto N(b,V) - N(a,V), \quad 0 < a \le b, V \in \mathcal{B}(\mathbb{R}_0)$$

The expression in equation (2.27) is called the *jump measure* of L. Its differential form is denoted by $N(dt,dx), t > 0, x \in \mathbb{R}_0$. Hence the respective Lévy measure ν of L is defined by

$$\nu(V) := \mathbb{E}(N(1,V)), \quad V \in \mathcal{B}(\mathbb{R}_0) \tag{2.28}$$

Note that ν need not to be a finite measure but it always satisfies

$$\int_{\mathbb{R}_0} (1 \wedge x^2)\nu(dx) < \infty$$

Since the sum of small jump $\sum_{s\le t}\Delta L(s)\mathbb{I}\{|\Delta L(s)|\le 1\}$ does not converge in general (infinite many small jumps), so we need to force this sum to converge by compensating it. Hence we define the *compensated jump measure* $\tilde{N}$ as

$$\tilde{N}(dt,dx) := N(dt,dx) - \nu(dx)dt \tag{2.29}$$

It is known that every Lévy process can be represented as a compound Poisson process in some way. As we have seen previously, every Lévy process can be represented with a continuation of a constant trend, a scaled Wiener process, and some jump process with stationary independent increments, independent of the Wiener process [78; 66]. If the number of jumps in every finite interval is almost surely finite, then the pure-jump component is a compound Poisson process. This criterion allows us to construct geometric Lévy process based models (as we will see shortly), not only ones similar to jump-diffusion, but also processes with an infinite number of jumps in finite intervals. We now formalise this idea into the following

Theorem 2.14 (Lévy-Itô decomposition theorem)**.** *Let $L=\{L(t),t\ge 0\}$ be a Lévy process and ν its Lévy measure. L can be decomposed into*

$$L = L^{(0)} + L^{(1)} + L^{(2)} + L^{(3)}$$

where $L^{(0)}$ is an affine (linear) function, $L^{(1)}$ is a scaled Wiener process, $L^{(2)}$ is a compound Poisson process with jump size smaller than 1. $L^{(3)}$ is a Lévy process with jump sizes smaller than 1. The processes $L^{(i)}$ are independent of each other. More precisely,

$$L(t) = \theta t + \sigma W(t) + X^{cpp}(t) + \lim_{\epsilon\to 0}\tilde{X}^{\epsilon}(t)$$

where

$$X^{cpp}(t) = \int_0^t\int_{\{|x|\ge 1\}} xN(ds,dx) = \sum_{s\in[0,t]}^{|\Delta X(s)|\ge 1}\Delta X(s) = \sum_{s\in(0,t]}\Delta X(s)\mathbb{I}\{|\Delta X(s)|\ge 1\} \tag{2.30}$$

$$\tilde{X}^{\epsilon}(t) = \int_0^t\int_{\epsilon\le|x|<1} x\tilde{N}(ds,dx) = \sum_{s\in(0,t]}^{\epsilon\le|\Delta X(s)|<1}\Delta X(s) = \sum_{s\in[0,t]}\Delta X(s)\mathbb{I}\{\epsilon\le|\Delta X(s)|<1\} \tag{2.31}$$

Then L, admits the following integral representation in terms of the jump measure $N(ds,dx)$

$$L(t) = \theta t + \sigma W(t) + \sum_{s\in(0,t]}^{|\Delta X(s)|\ge 1}\Delta X(s) + \sum_{s\in(0,t]}^{\epsilon\le|\Delta X(s)|<1}\Delta X(s) \tag{2.32a}$$

$$L(t) = \theta t + \sigma W_t + \int_0^t\int_{|x|\ge 1} xN(ds,dx) + \int_0^t\int_{|x|<1} x\tilde{N}(ds,dx) \tag{2.32b}$$

for some constants $\theta,\sigma\in\mathbb{R}$. Here $W=\{W(t),t\ge 0\}$, $W(0)=0$, is a standard Wiener process.

The process X^{cpp} is a compound Poisson process, the process $\tilde{X}^{\epsilon}$ is a compensated compound Poisson process (See Appendix A, proposition A.6 for definition), which is a martingale. With the aid of the notion of Jump measure and notation introduce in equation (2.28), one can express equation (2.32a) as equation (2.32b). Notice that $\int_0^t\int_{\{\epsilon\le|x|<1\}} xN(ds,dx)$ and $\int_0^t\int_{\{\epsilon\le|x|<1\}} x\nu(dx)ds$ are well defined outside 0. However, these quantities do not converge as ϵ tends to 0. Intuitively, the term $\int_0^t\int_{|x|<1} x\tilde{N}(ds,dx)$ could be thought of as small jumps that dictate the day-to-day fluctuation of a stock price. On the other hand, the large jumps are captured by the term $\int_0^t\int_{|x|\ge 1} xN(ds,dx)$ that describe large stock price movement caused by extreme market shock, such as terrorist attack and earthquakes. In particular, we see that if the Lévy process has continuous trajectories, then it is of the form which is familiar to us

$$L(t) = \theta t + \sigma W(t), \quad t \geq 0$$

It could be proven if

$$\mathbb{E}(|L(t)|^p) < \infty \quad \text{for some } p \geq 1$$

then

$$\int_{|x|\geq 1} |x|^p \nu(dx) < \infty.$$

In particular, take $p = 2$, by assuming

$$\mathbb{E}(|L(t)|^2) < \infty \tag{2.33}$$

We have

$$\int_{|x|\geq 1} |x|^2 \nu(dx) < \infty.$$

Proposition 2.15. *Let $\{L(t), t \geq 0\}$ be a compound Poisson process with intensity 1. The jump measure N_L is a Poisson random measure on $\mathbb{R} \times [0, \infty]$ with intensity measure $N(dx \times dt) = \nu(dx)dt$*

Apply proposition 2.15, equation (2.32b) becomes the celebrated Itô decomposition of Lévy process.

$$L(t) = \theta t + \sigma W(t) + \int_0^t \int_{|x|\geq 1} x\nu(dx)ds + \int_0^t \int_{|x|<1} x\tilde{N}(ds, dx) \tag{2.34}$$

When $\sigma = 0$, the diffusion term vanishes and we obtain a pure jump Lévy process.
From now on, we assume equation (2.33) holds and the Lévy process X is represented as equation (2.34). Note that one can express equation (2.34) in a stochastic integral form (so-called Itô Lévy process)

$$L(t) = L(0) + \int_0^t \theta ds + \int_0^t \sigma dW(s) + \int_0^t \int_{|x|<1} x\tilde{N}(ds, dx) + \int_0^t \int_{|x|\geq 1} x\nu(dx)ds \tag{2.35}$$

Apply proposition 2.15 and equation (2.35) becomes

$$L(t) = L(0) + \int_0^t \theta ds + \int_0^t \sigma dW(s) + \int_0^t \int_{|x|<1} x\tilde{N}(ds, dx) + \int_0^t \int_{|x|\geq 1} N(dx, ds) \tag{2.36}$$

And in the equivalent short-hand differential notation is

$$dL(t) = \theta dt + \sigma dW(t) + \int_{|x|<1} x\tilde{N}(dt, dx) + \int_{|x|\geq 1} xN(dt, ds) \tag{2.37}$$

The Lévy process is semi-martingale, hence the sum of squared jumps is finite, i.e.

$$\sum_{0<s<t} (\Delta L(s))^2 < \infty$$

The following result is the basic building block in studying stochastic calculus for Lévy processes. This is the Itô Lévy formulae. We present two versions of this formulae. The first version stands from the first principle, [15], while the the second version makes use of random measures [20].

Theorem 2.16 (Lévy Itô formula 1)**.** *Let $L = \{L(t), t \geq 0\}$ be the Itô-Lévy process with given by equation (2.36) and let $f : (0, \infty) \times \mathbb{R} \longmapsto \mathbb{R}$ be a $C^{2,1}$ function and define*

$$Y(t) := f(t, L(t)), \quad t \geq 0$$

Then the process $Y = \{Y(t), t \geq 0\}$ is also an Itô Lévy process and its solution is given by

$$\begin{aligned}
Y(t) &= f(t, L(t)) \\
&= f(0, L(0)) + \int_0^t [f_s(s, L(s)) + f_x(s, L(s-))\theta + \frac{1}{2} f_{xx}(s, L(s))\sigma^2] ds \\
&+ \int_0^t f_x(s, L(s))\sigma dW(s) \\
&+ \sum_{s \in (0,t]}^{\Delta L(s) \neq 0} \{f(s, L(s-) + \Delta L(s)) - f(s, L(s-)) - \Delta L(s) f_x(s, L(s-))\}
\end{aligned} \tag{2.38}$$

and the respective differential form is given by

$$\begin{aligned}
dY(t) &= [f_t(t, L(t)) + f_x(t, L(t-))\theta + \frac{1}{2} f_{xx}(t, L(t))\sigma^2] dt \\
&+ f_x(t, L(t))\sigma dW(t) + \{f(t, L(t-) + \Delta L(t)) - f(t, L(t-)) - \Delta L(t) f_x(t, L(t-))\}
\end{aligned} \tag{2.39}$$

With the aid of the notion of jump measure, the first version of Lévy Itô formula can be written into a more elegant way.

Theorem 2.17 (Lévy Itô formula 2). *Let $L = \{L(t), t \geq 0\}$ be the Itô-Lévy process with given by equation (2.36) and let $f : (0, \infty) \times \mathbb{R} \longmapsto \mathbb{R}$ be a $C^{2,1}$ function and define*

$$Y(t) := f(t, L(t)), \quad t \geq 0$$

Then the process $Y = \{Y(t), t \geq 0\}$ is also an Itô Lévy process and its solution is given by

$$\begin{aligned}
Y(t) &= f(t, L(t)) \\
&= f(0, L(0)) + \int_0^t [f_s(s, L(s)) + f_x(s, L(s-))\theta + \frac{1}{2} f_{xx}(s, L(s))\sigma^2] ds \\
&+ \int_0^t \int_{\mathbb{R}_0} f_x(s, L(s))\sigma dW(s) + \int_0^t (f(s, x + L(s-)) - f(s, L(s-)))\tilde{N}_L(ds, dx) \\
&+ \int_0^t \int_{\mathbb{R}_0} (f(s, x + L(s)) - f(s, L(s)) - x f_x(s, L(s)))\nu(dx) ds
\end{aligned} \tag{2.40}$$

and the respective differential form is given by

$$\begin{aligned}
dY(t) &= [f_t(t, L(t)) + f_x(t, L(t-))\theta + \frac{1}{2} f_{xx}(t, L(t))\sigma^2] dt \\
&+ f_x(t, L(t))\sigma dW_t + \int_{\mathbb{R}_0} (f(t, x + L(t-)) - f(t, L(t-)))\tilde{N}(dt, dx) \\
&+ \int_{\mathbb{R}_0} (t, f(L(t) + x) - f(t, L(t)) - x f_x(t, L(t)))\nu(dx) dt
\end{aligned} \tag{2.41}$$

The two representations of the Lévy Itô formulae are equivalent. The reason is that Lévy process is a semi-martingale that possesses the following property:

$$\sum_{s \in (0,t]} (\Delta X(s))^2 < \infty$$

In equation (2.38), due to the semimartingale property, if function $f(\cdot)$ is a twice integrable in x, then,

$$|f(s, X(s-) + \Delta X(s)) - f(s, X(s-)) - \Delta X(s) f_x(s, X(s-))| \leq c(\Delta X(s))^2$$

Hence, the series

$$\sum_{s\in(0,t]} f(s, X(s-) + \Delta X(s)) - f(s, X(s-) - \Delta X(s) f_x(X(s-)))$$

converges.

2.7 Geometric Lévy processes

Starting from the well-known geometric Brownian motion with the following explicit form:

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

It is quite natural to replace the Wiener process with a more general Lévy process $L(t)$ and consider the process

$$S(t) = S(0)e^{L(t)}$$

The processes of this kind are so-called geometric Lévy processes and can be viewed as a generalisation of the Black Scholes model. As we will see, the study of exponential and stochastic exponential is essential in the sense that it allows us to construct the model of interest that describes processes driven by market information. We start with the ordinary exponential of a Lévy process. In the classical diffusion model, the evolution of the stock price is described by the exponential of a Wiener process with drift:

$$S(t) = S(0)e^{L^0(t)} \tag{2.42}$$

where

$$L^0(t) = mt + \sigma W(t)$$
$$dL^0(t) = mdt + \sigma dW(t)$$

Apply Itô formula to $S(t)$, we have, Then

$$\begin{aligned} dS(t) &= S(t)(mdt + \sigma dW(t) + \frac{1}{2}d[L^0, L^0](t)) \\ &= S(t)(mdt + \sigma dW(t) + \frac{1}{2}\sigma^2 dt) \\ &= S(t)((m + \frac{1}{2}\sigma^2)dt + \sigma dW(t)) \end{aligned} \tag{2.43}$$

Integrating both sides of equation (2.43),

$$\begin{aligned} \int_0^t dS(u) &= S(t) - S(0) + \int_0^t S(u)(m + \frac{1}{2}\sigma^2)du + \sigma \int_0^t S(u)dW(u) \\ \implies S(t) &= S(0) + \int_0^t S(u)(m + \frac{\sigma^2}{2})du + \sigma dW(u) \\ &= S(0) + \int_0^t S(u)dL^1(u) \end{aligned} \tag{2.44}$$

where $L^1(u) = (m + \frac{\sigma^2}{2})t + \sigma W(t)$ is another Wiener process with drift. Replacing $L^0(t)$ in the process $S(t)$ itself (2.42) , one can obtain geometric Lévy models

$$S(t) = S(0)e^{L(t)}, \text{ where } L(t) \text{ is a Lévy process}$$

Replacing $L^1(t)$ by a Lévy process in equation (2.44) we obtain the stochastic exponential.[8] We see that this process has a solution of the form

$$S(t) = S(0) + \int_0^t S(u-)dL(u)$$

that satisfies a SDE of the same form as the classical ODE for the exponential function.

2.7.1 Ordinary Exponential for Lévy process

Proposition 2.18. *Let $\Delta X(t) = X(t) - X(t-)$ be the jump measure of a Lévy process $X(t)$, N_X be the random measure of X as defined in equation (2.27), the following holds,*

$$\sum_{0\leq s\leq t;\Delta X(s)} \{(f(X(s-) + \Delta X(s)) - f(X(s-))\} = \int_0^t \int_{\mathbb{R}} \{(f(X(s-) + y - f(X(s-))\} N_X(ds, dy)$$

Let $\{L(t), t \geq 0\}$ represents a Lévy process with jump measure N_L as follows:

$$L(t) = \theta t + \sigma W(t) + \sum_{s\in[0,t]} \Delta X(s) \tag{2.45}$$

In differential form, equation (2.45) can be expressed as follows

$$dL(t) = \theta dt + \sigma dW(t) + \Delta X(t)$$

Applying Itô lemma (2.38 and 2.40) to $Y_t = e^{L_t}$. We obtain,

$$\begin{aligned}
Y(t) &= f(t, L(t)) \\
&= f(0, L(0)) + \int_0^t [f_s(s, L(s)) + f_x(s, L(s))\theta + \frac{1}{2} f_{xx}(s, L(s))\sigma^2] ds \\
&+ \int_0^t f_x(s, L(s))\sigma dW(t) \\
&+ \sum_{s\in[0,t]}^{\Delta L(s)\neq 0} \{f(s, L(s-) + \Delta L(s)) - f(s, L(s-)) - \Delta L(s) f_x(s, L(s-))\} \\
&= 1 + \int_0^t Y(s)(1 + \theta + \frac{1}{2}\sigma^2) ds + \int_0^t Y(s)\sigma dW(t) \\
&+ \sum_{s\in[0,t]}^{\Delta L(s)\neq 0} \{e^{L(s-)+\Delta L(s)} - e^{L(s-)} - \Delta L(s) e^{L(s-)}\} \\
&= 1 + \int_0^t Y(s-)(1 + \theta + \frac{1}{2}\sigma^2) ds + \int_0^t Y(s-)\sigma dW(t) \\
&+ \int_0^t \int_{\mathbb{R}_0} Y(s(-))(e^x - 1)\tilde{N}_L(ds, dx) + \int_0^t \int_{\mathbb{R}_0} (e^x - 1 - x\mathbb{I}\{|z| \leq 1\})\nu(dx)
\end{aligned}$$

In some Mathematical Finance applications, i.e. when constructing a trading strategy, it is often useful to split the stochastic integral like $Y(t)$ into a deterministic part and a martingale part. To summarise, we have the following proposition.

Proposition 2.19 (Exponential of a Lévy process). *Let $\{L(t), t \geq 0\}$ be a Lévy process with generating a triplet (θ, σ^2, ν) satisfying*

$$\int_{|y|\geq 1} e^y \nu(dy) < \infty$$

[8] The stochastic exponential is also called Doléan-Dade exponential , which is first introduced by French female mathematician Doléan-Dade [21].

Then $Y(t) = \exp(L(t))$ is a semimartingale that can be decomposed $Y(t) = M(t) + A(t)$ where the martingale part is given by

$$M(t) = 1 + \int_0^t Y(s-)\sigma dW(s) + \int_{[0,t]\times\mathbb{R}} Y(s-)(e^x - 1)\tilde{N}_L(ds,dx) \tag{2.46}$$

and the continuous finite variation drift part is given be

$$A(t) = \int_0^t Y(s-)(1 + \theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbb{I}\{|x| \le 1\})\nu(dx))ds \tag{2.47}$$

$\{Y(t)\}$ is a martingale if and only if

$$1 + \theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbb{I}\{|x| \le 1\})\nu(dx) = 0 \tag{2.48}$$

2.7.2 Stochastic Exponential and Geometric Lévy process

Let $L = \{L(t), t \ge 0\}$ be a real-valued Lévy process with a generating triplet (θ, σ^2, ν), $L(0) = 0$. There exists a unique càdlàg process $Y = \{Y(t), t \ge 0\}$ such that

$$dY(t) = Y(t-)dL(t), Y(0) = 1 \tag{2.49}$$

Y is given by

$$Y(t) = e^{L(t) - \frac{1}{2}[L,L](t)} \prod_{0 \le s \le t} (1 + \Delta L(s))e^{-\Delta L(s) + \frac{1}{2}(\Delta L(s))^2} \tag{2.50}$$

or

$$Y(t) = e^{L(t) - \frac{1}{2}[L,L](t)^c} \prod_{0 \le s \le t} (1 + \Delta L(s))e^{-\Delta L(s)} \tag{2.51}$$

Since $[L,L](t) = [L,L]^c(t) + \sum_{0 \le s \le t} (\Delta L(s))^2$

The Doléan exponential[9] of L is denoted by $\mathcal{E}(L)$. Equation (2.51) is used for the rest of this thesis.

Proof. The proof of uniqueness is omitted (See Lipster & Shiryayev (1989) [45] for details). Let

$$U(t) = L(t) - \frac{1}{2}[L,L]^c(t), \; V(t) = \prod_{s \le t}(1 + \Delta L(s))e^{-\Delta L(s)}$$

Step 1 Show the infinite product $V(t)$ converges. In other words, we prove $V(t)$ is well-defined and is of finite variation.

$$V(t) = \underbrace{\prod_{s \in (0,t]} (1 + \Delta L(s))\mathbb{I}\{|\Delta L(s)| < 1/2\}e^{-\Delta L(s)}}_{V^{(1)}(t)} \underbrace{\prod_{s \in (0,t]} (1 + \Delta L(s))\mathbb{I}\{|\Delta L(s)| > 1/2\}e^{-\Delta L(s)}}_{V^{(2)}(t)} \tag{2.52}$$

Although the product is taken $s \le t$, there are only a finite number of s such that $|\Delta L(s)| \ge 1/2$ on each compact interval[10]. In other words, the second product $V^{(2)}$ contains a finite number of factors and so

[9] This definition can be found in Appendix A under Mathematic Preliminary section
[10] A compact interval on $\mathbb{R}$ is simply a closed interval on $\mathbb{R}$.

it is of finite variation. Thus it suffices to show the first product $V^{(1)}(t)$ is of finite variation. Taking the logarithm of $V^1(t)$ we have

$$\ln(V^1(t)) = \sum_{s\leq t}\{(\ln(1+\Delta L(s)\mathbb{I}\{|\Delta L(s)| \leq 1/2\}) - \Delta L(s)\mathbb{I}\{|\Delta L(s)| \leq 1/2\}\}$$

This series converge absolutely and almost surely since

$$\sum_{s\in(s,t]} (\Delta L(s)\mathbb{I}\{|\Delta L(s)| < 1/2\})^2 \leq [L,L](t) < \infty \text{ a.s.}$$

because $|\ln(1+x) - x| \leq x^2$ when $|x| < 1/2$. Clearly, $\ln(V^1(t))$ is a process with finite variation sample paths. This ensures $V(t)$ exist and has trajectories of finite variation.

Step 2 Show $Y(t)$ is a solution. Set $\hat{L}(t) = L(t) - \frac{1}{2}[L,L]^c(t)$ and let $f(x,y) = ye^x$. Then $Y(t) = f(\hat{U}(t), V(t))$, where U, V are defined previously. Now apply Itô formula we have

$$\begin{aligned}
Y(t) &= 1 + \int_0^t Y(s-)dU(s) + \int_0^t e^{U(s-)}dV(s) + \frac{1}{2}\int_0^t Y(s-)d[U,U]^c(s) \\
&\quad + \sum_{s\in(0,t]} (Y(s) - Y(s-) - Y(s-)\Delta U(s) - e^{U(s-)}\Delta V(s)) \\
&= \int_0^t Y(s-)dL(s) - \frac{1}{2}\int_0^t Y(s-)d[L,L]^c(s) + \int_0^t e^{U(s-)}dV(s) \\
&\quad + \frac{1}{2}\int_0^t Y(s-)d[L,L]^c(s) + \sum_{s\in(0,t]} (Y(s) - Y(s-) - Y(s-)\Delta U(s) - e^{U(s)} * \Delta V(s))
\end{aligned} \tag{2.53}$$

since $[U,V]^c = [V,V]^c = 0$. S is a pure jump process. Hence $\int_0^t e^{U(s-)}dS(s) = \sum_{s\in(0,t]} e^{U(s-)}\Delta V(s)$. Also $Y(s) = Y(s-)(1+\Delta L(s))$, and $Y(s-)\Delta U(s) = Y(s-)\Delta L(s)$, so the last sum in equation (2.53) becomes

$$\sum_{s\in(0,t]} (Y(s-)(+\Delta L(s))) - Y(s-) - Y(s-)\Delta L(s) - e^{U(s-)}$$

After some cancellation of terms we have

$$Y(t) = 1 + \int_0^t Y(s-)dL(s)$$

□

Y is called a stochastic exponential or Doléan exponential of L and denoted by $Y = \mathcal{E}(L)$. Note that, one could define it for any arbitrary semimartingale that not necessarily a Lévy process. However, it should be noted that the proof does not use the independent or stationary increment argument which we usually use for Lévy processes.

Equation (2.50) gives a general formula for $\mathcal{E}(L)$. The formulae simplifies when the Lévy process L is continuous. Indeed, suppose L is a continuous semimartingale with $L(0) = 0$. Then

$$\mathcal{E}(L)(t) = \exp\{L(t) - \frac{1}{2}[L,L](t)\}$$

Where the semimartingale L is a transformation of a Wiener process $W = \{W(t), t \geq 0\}$, scaled by some constant σ. Since σW has no jump we have

$$\sigma W(t) = \exp\{\lambda W(t) - \frac{\sigma^2}{2}[W,W](t)\} = \exp\{\sigma W(t) - \frac{\sigma^2}{2}t\}$$

Furthermore, since $\mathcal{E}(\sigma W)(t) = 1 + \sigma \int_0^t \mathcal{E}(\sigma W)(s-)dW(s)$, we see that $\mathcal{E}(\sigma W)(t) = e^{\sigma W(t) - \frac{\sigma^2}{2}t}$ is a continuous martingale. Some authors prefer to use the term "exponential martingale". The process $\mathcal{E}(\sigma W)$ is in fact the well-known *geometric Brownian motion*.

2.7.3 Relation between ordinary and stochastic exponential

It is clear from the previous result that the ordinary exponential and the stochastic exponential are essentially different: they do not correspond to the same stochastic process. Observe from equation (2.50), the stochastic exponential, $\mathcal{E}(L)$ is strictly positive if and only if the jump of L, i.e. $\Delta L > -1$. In contrast, the ordinary exponential, $\exp^L$ is clearly a strictly positive process. Hence, naturally, one may think which of the two processes is more suitable to model stock prices or returns. Remarkably Goll and Kallen [35] shows that the two approaches (modelling via an ordinary exponential and via a stochastic exponential) are equivalent. If $Y > 0$ is the stochastic exponential of a Lévy process then it is also the ordinary exponential of another Lévy process. The reverse is also true: If there is some process X which is the ordinary exponential of a Lévy process Y, then there exist another Lévy process Z such that the Lévy process X is the stochastic exponential of Z. Therefore, the two different operations, although produce different objects when applied to the same Lévy process, end up giving us the same class of positive processes. In continuous time, the previous argument is quite intuitive. Recall the previous example $L(t) = \sigma W$ and the ordinary exponential and the stochastic exponential are given respectively by

$$Y^1(t) = e^{\sigma W(t)} \text{ and } Y^2(t) = e^{\sigma W - \frac{\sigma^2}{2}t}$$

We see that the stochastic exponential of a Lévy process (the Wiener process with constant coefficient) is indeed the ordinary exponential of another Lévy process (the Wiener process with constant coefficient and affine drift). In short, the idea is: If some $Y > 0$ is the stochastic exponential of a Lévy process L, then it is also the ordinary exponential of another Lévy process $\tilde{L}$. More precisely, we have the following [15]:

Proposition 2.20 (Relation between ordinary and stochastic exponential)**.**

1. *Let* $= \{L(t), t \geq 0\}$ *be a real valued Lévy process with a generating triplet* (θ, σ^2, ν) *and* $Y = \mathcal{E}(L)$ *its stochastic exponential. If* $L > 0$ *a.s. then there exist another Lévy process* $\{\tilde{L}_t, t \geq 0\}$ *such that* $Y_t = e^{\tilde{L}_t}$ *where*

$$\tilde{L}_t = \ln Y_t = L_t - \frac{\sigma^2 t}{2} + \sum_{s \in (0,t]} \ln(1 + \Delta L_s) - \Delta L_s \tag{2.54}$$

Its generating triplet $(\tilde{\theta}, \tilde{\sigma}^2, \tilde{\nu})$ *is given by:*

$$\begin{aligned} \tilde{\sigma} &= \sigma, \\ \tilde{\nu}(A) &= \nu(x : \ln(1+x) \in A) = \int \mathbb{I}_A(\ln(1+x))\nu(dx), \\ \tilde{\theta} &= \theta - \frac{\sigma^2}{2} + \int \nu(dx)\ln(1+x)\mathbb{I}_{[-1,1]}(\ln(1+x)) - x\mathbb{I}_{[-1,1]}(x) \end{aligned} \tag{2.55}$$

2. *Let* $\tilde{L}, t \geq 0$ *be a real valued Lévy process with generating triplet* $(\tilde{\theta}, \tilde{\sigma}, \tilde{\nu})$ *and* $S_t = \exp(L_t)$ *its exponential. Then there exists a Lévy process* $L, t \geq 0$ *such that* S *is the stochastic exponential of* $L : S = \mathcal{E}(L)$ *where*

$$L_t = \tilde{L}_t + \frac{\sigma^2 t}{2} + \sum_{s \in (0,t]} (e^{\Delta L(s)} - 1 - \Delta L(s)) \tag{2.56}$$

Its generating triplet (θ, σ^2, ν) *of* L *is given by:*

$$\sigma = \tilde{\sigma},$$
$$\nu(A) = \tilde{\nu}(x : e^x - 1 \in A) = \int \mathbb{I}_A(e^x - 1)\tilde{\nu}(dx),$$
$$\theta = \tilde{\theta} + \frac{\tilde{\sigma}^2}{2} + \int \tilde{\nu}(dx)(e^x - 1)\mathbb{I}_{[-1,1]}(\ln(1+x)) - x\mathbb{I}_{[-1,1]}(x) \tag{2.57}$$

Proof. See [15]. □

A very important property of the stochastic exponential is the martingale invariance property. Basically it tells us that any stochastic exponential of a Lévy process is again a martingale. Nevertheless, it is formalised as follows:

Proposition 2.21. *If* $(X)_{t\geq 0}$ *is a Lévy process and a martingale, then its stochastic exponential* $Z = \mathcal{E}(X)$ *is also a martingale.*

Proof. See [15]. □

Example 2.5 (Generalized geometric Lévy process)**.** [20] Consider the one-dimensional stochastic differential equation for the cáglád process $S = \{S(t), t \geq 0\}$:

$$\begin{cases} dS(t) = S(t-)[\theta dt + \sigma dW(t) + \int_{\mathbb{R}_0} x\tilde{N}(dt, dx)], \quad t > 0 \\ S(0) = S_0 > 0 \end{cases}$$

Assuming all the technical conditions hold, we claim that the solution of example 2.5 is

$$S(t) = S_0 \exp\{X(t)\}, \quad t \geq 0 \tag{2.58}$$

where

$$\begin{aligned} X(t) = \int_0^t (\theta - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0} (\log(1+x) - x)\nu(dx))ds \\ \int_0^t \beta dW(s) + \int_0^t \int_{\mathbb{R}_0} \log(1+x)\tilde{N}(ds, dx) \end{aligned} \tag{2.59}$$

To visualise this we apply the one-dimensional Itô formula (2.41) to $Y(t) = f(t, X(t)), t \geq 0$, with $f(t,x) = S_0 e^x$ and $X(t)$, as given in (2.59). Then we obtain

$$\begin{aligned} dY(t) &= S_0 e^{X(t)}[(\theta - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0} [\log(1+x) - x]\nu(dx))dt] + \beta dW(t) \\ &+ S_0 e^{X(t)}\frac{1}{2}\sigma^2 dt + \int_{\mathbb{R}_0} S_0[e^{X(t)+\log(1+x)} - e^{X(t)} - e^{X(t)}\log(1+x)]\nu(dx)dt \\ &\int_{\mathbb{R}_0} S_0(e^{X(t-)+\log(1+x)} - e^{X(t-)})\tilde{N}(dt, dx) \\ &= Y(t-)(\theta dt + \sigma dW(t) + \int_{\mathbb{R}_0} x\tilde{N}(dt, dx)) \end{aligned} \tag{2.60}$$

as required.

2.7.4 Change of measure and absolute continuity for Lévy processes

The Girsanov theorem is a fundamental concept in the general theory of stochastic analysis. It also has important applications, for example in Mathematical Finance. When pricing a contingent claim traded in the financial market, the probability measure we use is usually different from the statistical measure we observe. In probability theory, Girsanov theorem tells us how stochastic processes change under change in measure. Therefore, it is the key theorem for the classical Black-Scholes model in connecting the physical measure with the risk-neutral measure. Girsanov theorem simply says that if we change the *drift* coefficient of a given Itô process with a nondegenerate drifted diffusion, then the law of the process will not change in its form. Indeed, the law of the new process will be absolutely continuous w.r.t. the law of the original process and the Radon-Nikodym derivative can be computed explicitly [63]. In this section we first recall some basic facts on absolute continuity and equivalence of probability measure, then describe the change of measure in the case of Brownian motion, finally we present the analogous Girsanov theorem for Lévy process.

Definition 2.13. Given two probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined on the same σ-algelbra $\mathcal{F}$,

i $\mathbb{P}$ is said to be absolutely continuous with respect to $\mathbb{Q}$, or dominated by $\mathbb{Q}$ if $\mathbb{P}(A) = 0$ for every set A for which $\mathbb{Q}(A) = 0$, for all $A \in \mathcal{F}$.

ii if $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ and $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, then we call $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, denoted $\mathbb{P} \sim \mathbb{Q}$

In the context of financial modelling, the equivalence of measures is important. This is due to the fact that equivalent measures have the same a.s. and null sets, hence by changing between measures, we do not alter the possible states in the economy, we only alter the probabilities assigned to each state. By the First Theorem of Asset Pricing, this ensures the fairness of prices under changes of measures and changes of numeraires.

Theorem 2.22. *Let $W(t)$ be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and let $\lambda(t)$ be a adapted process*[11] *satisfying Novikov's condition,*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T |\lambda_t|^2 dt\right)\right) < \infty$$

Moreover let $\mathcal{E}_T(\cdot)$ be the Doléan exponential. Define an equivalent measure $\mathbb{Q}$ by

$$\begin{aligned} \eta_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}(t)} &= \mathcal{E}_t\left(\int_0^{\cdot} \lambda_s dW_s\right) \quad \mathbb{P} \quad a.s. \\ &= \exp\left(\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t |\lambda_s|^2 ds\right) \quad \forall t \in [0, T] \end{aligned}$$

as an exponential martingale *with respect to the natural Brownian filtration $\mathbb{F}$ under the probability measure $\mathbb{P}$*
The relation

$$\mathbb{Q}(A) = \int_A^{\cdot} \eta_T(\omega) d\mathbb{P}(\omega), \Longleftrightarrow \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \eta_T$$

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \eta_T$$

defines a probability measure $\mathbb{Q}$ on $\mathcal{F}$ which is equivalent to $\mathbb{P}$. Under the probability measure $\mathbb{Q}$, the

[11] λ_t must satisfy condition to ensure the Radon Nikodym derivative is a martingale

process

$$W_t^{\mathbb{Q}} = W_t - \int_0^t \lambda_s ds, \quad t \in [0,T] \quad \text{is a standard B.M. under } \mathbb{Q}$$

i.e.

$$dW_t^{\mathbb{Q}} = dW_t - \lambda_t dt$$

is a standard Brownian motion under measure $\mathbb{Q}$.
The process $W^{\mathbb{Q}}$ *is adapted to the filtration* $\mathbb{F}$.
The probability measure $\mathbb{P}$ *is called an* equivalent martingale measure.

The Girsanov theorem for Brownian motion simply states that by changing the drift of a given Brownian motion, one can find an equivalent measure under which the new process is again a Brownian motion. The law of the new Brownian motion will be absolute continuous w.r.t. the law of the original Brownian motion, and the Radom-Nikodym derivative can be computed explicitly. The measure change results between two Lévy processes [15] are similar to what we have just seen. We consider the classic Girsanov theorem as a special case of the following theorem.

Theorem 2.23 (Generalised Girsanov)**.** *Let* $X = \{X(t), t \geq 0\}$ *be a real-valued Lévy process on* $(\Omega, \mathcal{F}, \mathbb{Q})$ *and on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with respective generating triplets* $(\theta_{\mathbb{Q}}, \sigma_{\mathbb{Q}}^2, \nu_{\mathbb{Q}})$ *and* $(\theta_{\mathbb{P}}, \sigma_{\mathbb{P}}^2, \nu_{\mathbb{P}})$. *Then* $\mathbb{Q}|_{\mathcal{F}(t)} << \mathbb{P}|_{\mathcal{F}(t)}$ *for all* $t \in [0, T_\infty]$ *iff the following conditions hold:*

1. *They have the same diffusion component:* $\sigma_{\mathbb{Q}}^2 = \sigma_{\mathbb{P}}^2 := \sigma^2$

2. $\nu_{\mathbb{Q}} \ll \nu_{\mathbb{P}}$

3. *The Lévy measure are equivalent with*

$$\int_{-\infty}^{\infty} (e^{\eta(x)/2} - 1)^2 \nu_{\mathbb{P}}(dx) = \int_{-\infty}^{\infty} (\sqrt{\eta(x)} - 1)\nu_{\mathbb{P}}(dx) < \infty$$

4. *If* $\sigma = 0$ *then we must in addition have* $\theta_{\mathbb{Q}} - \theta_{\mathbb{P}} = \int_{|x|\leq 1} x(\eta(x)-1)\nu_{\mathbb{P}}(dx)$ *Furthermore, when* $\mathbb{P}$ *and* $\mathbb{Q}$ *are equivalent, the Radon-Nikodym derivative is*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{U(t)}$$

where

$$U(t) = \lambda L^c(t) - \frac{\lambda^2\sigma^2 t}{2} - \lambda\theta t + \lim_{\epsilon \searrow 0}\left(\sum_{s\leq t, |\delta L(s)|>\epsilon} \eta(\Delta L(s)) - t\int_{|x|>\epsilon} (e^{\eta(x)} - 1)\nu(dx)\right)$$

Note, $L^c(t)$ *is the continuous part of* $L(t), \lambda \in \mathbb{R}$ *is chosen such that*

$$\theta_{\mathbb{Q}} - \theta_{\mathbb{P}} - \int_{|x|\leq 1} x(\eta(x) - 1)\nu_{\mathbb{P}}(dx) = \sigma_{\mathbb{P}}^2\lambda \;\; \text{if } \sigma_{\mathbb{P}} > 0, \;\text{ and } \lambda = 0 \;\text{ if } \sigma = 0$$

$U(t)$ *is a Lévy process with the generating triplet* $(\theta_U, \sigma_U, \nu_U)$ *given by*

$$\sigma_U = \sigma_{\mathbb{P}}^2\lambda^2, \nu_U = \nu_{\mathbb{P}}\eta^{-1}\theta_U = -\frac{1}{2}\sigma_{\mathbb{P}}^2\lambda^2 - \int_{-\infty}^{\infty}(e^y - 1 - y\mathbb{I}_{\{|y|\leq 1\}})\nu\eta^{-1}(dy)$$

The above result suggests that the change of measure can be done in a more flexible manner while preserving the equivalence of measure under Lévy process. However, if the diffusion term is absent, then the drift is freely changed. Under such circumstances, one may change the distribution of the jump $\nu(dx)$. Hence changing measure under Lévy processes, one needs to find equivalent martingale measures (E.M.M.). An E.M.M. $\mathbb{Q}$ is an absolutely continuous probability measure w.r.t. the original measure $\mathbb{P}$ that makes the discounted price process a martingale. The pricing model being studied in this thesis: the Geometric Lévy process model is an incomplete market model, we recognise that the option pricing for such a model forces a move out of the traditional realm of arbitrage pricing into the domain of equilibrium pricing, since there exists an infinite number of martingale measures under this model by the Second Theorem of Asset Pricing. The choice of a suitable martingale measure is important for the purpose of pricing option based on arbitrage theory. But fortunately, our setting allows us to solve the pricing problem by utilising the classic version of Girsanov theorem. Nevertheless, the following are a list of candidates for choosing a martingale measure for option pricing [55].

- Minimal Martingale Measure (MMM) (Föllmer-Schweizer(1991)[29])
- Variance Optimal Martingale Measure (VOMM) (Schweizer(1995)[77])
- Mean Correcting Martingale Measure (MCMM)
- Esscher Martingale Measure (ESMM) (Gerber-Shiu(1994),B-D-E-S(1996)[33])
- Minimal Entropy Martingale Measure (MEMM) (Miyahara(1996)[54], Frittelli(2000)[30], Miyahara & Novikov (2002)[56])
- Utility Based Martingale Measure (U-MM)

2.7.5 Geometric Lévy based pricing model: Basic notions

Now we turn our focus to the stock price model of this thesis: the *geometric Lévy model.* First, we review the classical pricing model.
The famous Black-Scholes model

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

is a typical model for a complete market[12]. This model is an outstanding model based on its simplicity, but it is well-known that in the real world, the market completeness is not usually satisfied. There is abundant empirical evidence that financial price processes do not have Gaussian-distributed returns. For example, the phenomenon of *implied volatility smile* in option markets suggests that the risk-neutral returns are non-gaussian and leptokurtic. While the smile itself can be explained within a diffusion based model with continuous paths, the fact that it becomes much more pronounced for short maturity options clearly indicates the presence of jumps. From a market microstructure point of view, every price process is essentially discrete. A continuous process such as Brownian motion is used as a proxy for the real discrete observed process. Hence there is a need to introduce models to allow jump. Second, in diffusion-based models, the law of returns for shorter maturity becomes more or less Gaussian law,whereas in the real world, returns actually become less Gaussian as the time horizon becomes shorter. The third argument is that jump processes correspond to genuinely incomplete market, whereas all diffusion-based model are either complete or can be made complete with a small number of additional assets. Hence, we need a model that takes into account the weakness of the traditional model. The geometric Lévy model

[12] The market is said to be complete if every contingent claim is attainable. A more precise definition can be found in Appendix A, see A.8 the Second Theorem of Asset Pricing

is one of them. This model is well suited to an incomplete market. A great advantage of the geometric Lévy model is the mathematical tractability, which makes it possible to perform many computations analytically and to derive useful results in a simple manner. This leads to an explosion of literature on option pricing (including hedging) in geometric Lévy models in the late 90s and early 2000s, the literature contains hundreds of research papers and several monographs to date. Now, let's make this more precise.

Suppose that in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a geometric Lévy process (GLP) is obtained by replacing the Brownian motion with drift (continuous Lévy process) in the classical Black-Scholes model of an asset price, by a Jump-type Lévy process [55]:

$$S_t = S_0 e^{L_t} \tag{2.61}$$

where L_t is a Lévy process with a generating triplet $(\theta, \sigma^2, \nu(dx))$. The price process S_t has the following expression:

$$S_t = S_0 \mathcal{E}(\tilde{L})_t$$

where $\mathcal{E}(\tilde{L})_t$ is the Doléans-Dade exponential of $\tilde{L}_t$ and a generating triplet $\tilde{L}_t$, say $(\tilde{\theta}, \tilde{\sigma}^2, \tilde{\nu}(dx))$ is

$$\tilde{\theta} = \theta + \frac{1}{2}\sigma^2 + \int \left((e^x - 1)\mathbb{I}_{\{|e^x-1|\le 1\}} - x\mathbb{I}_{\{|x|<1\}}\right)\nu(dx) \tag{2.62}$$

$$\tilde{\sigma} = \sigma \tag{2.63}$$

$$\tilde{\nu}(dx) = (\nu \circ (e^x - 1)^{-1})(dx) \tag{2.64}$$

$$\text{that is} \quad \tilde{\nu}(A) = \int_A (e^x - 1)\nu(dx)$$

and the log returns $\ln(S_t/S_0) = L_t$ can be any Lévy process. This model fits quite well the empirical distribution of the asset returns. However, pricing of vanilla options under these models is not as simple as with diffusion-based models since the uniqueness of the equivalent martingale measure is not preserved in most of the Lévy models. Thus, Lévy financial models lead to incomplete markets[13] in which there are infinitely many equivalent martingale measures and perfect hedges are unattainable. To price options under these models, one needs to first choose a risk-neutral measure from the equivalent martingale measures available. Several methods are available including Esscher transform, minimal entropy measure and indifference pricing, which have already been mentioned.

By the First Fundamental Theorem of Asset Pricing (see appendix A, A.7 or [19]), a financial market is arbitrage free if and only if there exist at least one risk neutral measure that is equivalence to the original probability measure. In other words, there is no free lunch[14] in the financial market if we can find a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted price $e^{-rt}V_t$ of all assets are $\mathbb{Q}$-local martingales. $\mathbb{Q}$ is called a risk-neutral probability. Hence the absence of arbitrage implies the existence of probability $\mathbb{Q} \sim \mathbb{P}$, such that e^L is $\mathbb{Q}$-local martingale. The following result shows that if L is a Lévy process under P, one can always find an equivalence measure $\mathbb{Q}$, under which L is still a Lévy processes and e^L is a martingale.

Theorem 2.24 (Absence of arbitrage in Geometric Lévy models). *Let $L = \{L(t), t \ge 0\}$ be a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ with a generating triplet (θ, σ^2, ν). If the trajectories of L are not monotone, then there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that under $\mathbb{Q}$, L is a Lévy process and e^L is a martingale.*

[13] Incomplete markets refers to markets in which the number of Arrow Debreu securities is less than the number of states of nature

[14] This means no profit can be made without taking risk. For a mathematical formulation, see theorem A.9 in appendix, for a even more precise mathematical treatment, see [40].

$\mathbb{Q}$ *may be chosen in such a way that* $(L, \mathbb{Q})$ *will have a generating triplet* $(\theta_{\mathbb{Q}}, \sigma^2_{\mathbb{Q}}, \nu_{\mathbb{Q}})$

Remark 2.3. Set $\mathbb{Q} \sim \mathbb{P}$. It is important to note that a $\mathbb{P}$ Lévy process is not necessarily a $\mathbb{Q}$ Lévy process. The class of Lévy process is not stable under a change of measure.

Proof. See theorem 33.1 and 33.2 in Sato (1999) [74]. □

Risk-neutral Geometric Lévy models Geometric Lévy models of type equation (2.61), where L is a Lévy process with generating triplet (θ, σ^2, ν), satisfying the condition that $\{e^{L_t}, t \geq 0\}$ is a martingale, are called risk-neutral geometric Lévy models and can be parametrised by σ^2 and ν only: $\mathbb{Q} = \mathbb{Q}(\sigma^2, \nu)$. Under a risk-neutral probability $\mathbb{Q}(\sigma^2, \nu)$ we can evaluate call option prices as the discounted expected terminal payoffs:

$$C^{\mathbb{Q}(\sigma^2,\nu)}(T,K) = e^{-rT}\mathbb{E}^{\mathbb{Q}(\sigma^2,\nu)}((S(T)-K)^+) = e^{-rT}\mathbb{E}^{\mathbb{Q}(\sigma^2,\nu)}((S_0 e^{rT+L(T)} - K)^+) \tag{2.65}$$

In this thesis, we chose to work with geometric Lévy process of the form equation (2.61), where the Lévy process $L_t = \theta t + \sigma W_t + X_t^{\text{VG}}$. The rationale is the following.

First, we chose to work with Lévy processes based on its tractability. It is true that there are many models around (examples are stochastic volatility model, GARCH model), Brownian motion is simple, but not good enough. So we need another model. Models based on Lévy process are reasonably tractable. Computations that can be performed with Lévy process that cannot be performed under a more general class of stochastic processes.

Second, we would like to preserve the simplicity of geometric Brownian motion. Having noted that geometric Brownian motion is a special type of geometric Lévy process where the Lévy process is continuous, i.e. $L_t = \theta t + \sigma W_t$. To incorporate jumps, we replace the continuous Lévy process with the Jump-type Lévy process of the following form:

$$L_t = \theta t + \sigma W_t + X_t^{\text{VG}} \tag{2.66}$$

Third, the jumps are considered to be VG distributed. We were seeking a process that could best described the behaviours of the jumps in the trajectories of the VWAP based on the criterions in Madan and Seneta (1990) [48].

1. Long tailedness relative to the normal for daily returns, with returns over longer periods approaching normality;
2. Finite moments of at least low orders;
3. Consistency with an underlying, continuous-time stochastic process, with independent and stationary increments;
4. Extension to multivariate process with elliptical multivariate distributions that thereby maintain validity of the capital asset pricing model.

There are four candidate distributions in modelling the jump component: the stable distribution [27; 49], the Praetz t distribution [68], the compound Event model [69] and the VG distribution [48]. The stable distribution does not possess finite moments of any order, hence it fails criterion two and three. The Praetz t distribution fails criterion three as it is not possible to construct a stochastic process with the criterion 3 because the sum of independent t variable is not a t-variable. The compound Event model[15]

[15] This is a model based on compound Poisson process with Gaussian-distributed jump size

satisfies all four properties. If the compound Event model is used to model the jump component, then the Lévy process becomes the well known Merton *jump diffusions* of the following form:

$$L(t) = \theta t + \sigma W_t + X_t^{\text{cpp}}$$

In this model, is that the diffusion part σW_t is responsible for the usual fluctuations in the return series. The jump part $X_t^{\text{cpp}} = \sum_{k=1}^{N_t} \xi_k$ is a compound Poisson process with finite many jumps in every time interval is responsible for the rare events. ξ_k is assumed to be Gaussian distributed. Some authors favour this model as it is convenient for simulations and capable in capturing large jumps. Models of this type also perform quite well for the purposes of volatility smile interpolation (See chapter 13 in [15]). However, several weak points of these models are:

- Jumps are assumed to be rare events.
- Distribution of jump size is restricted to be Gaussian process.
- Close form probability density does not exist.
- Finite number of jumps in every time interval.

When considering incorporating jumps in the problem of VWAP options pricing, we expect a large number of small jumps with occasional large move. Such features of jumps cannot be found in a merton *jump diffusions*-type model. The jumps under such models are usually considered to be of low frequency and high severity. This is one of the reason decomposition of the form as in equation (2.66) is often used in non-life insurance to describe some particular types of claim processes [58]. Also, the jumps are not necessarily to be rare events, they shall be the drivers of the price movement. In addition, close form probability density is necessary to perform more computations. We were looking for a model that is less restrictive and more realistic. The VG model is a subclass of pure jump processes with finite variation and infinite activity, appears to be the a right candidate for our purpose. *First*, the assumption of jumps are rare events is relaxed in the sense that the price process moves essentially by jumps. *Second*, the distribution of jump size need not to be known in advance, it can arrive infinitely often in every time interval. This is because, the VG model arises by evaluating a Brownian motion with drift at independent random time given by a Gamma process. A gamma process is infinite divisible, this gives rise to a Lévy process with infinite number of jumps in every time interval. *Third*, close form probability density does exist, as we will discuss in Chapter 4. Hence, clearly, the VG process possesses all the four properties described in [48]. In fact, one can show that the VG process is a limit of a particular compound events models in which the arrival rate of the jumps approaches infinity.

Based on the discussion above, the VG process respects the intuition behind a Wiener process and a compound Poisson process for the purpose of modelling stock prices in the sense that it balances between the two processes. Implementation of a Geometric Lévy model with VG jumps can be viewed as a bridge between a classical financial model and an insurance model. Nevertheless, the idea could be made clearer when visualising the trajectories of VG process in Chapter 4.

2.7.6 Geometric Lévy based pricing model: Applications to Finance

2.7.6.1 Categories of Geometric Lévy process

Let's now review several geometric Lévy models in the existing financial literature. These models are used to describe the stock price evolution under both the historical (real-world) and the risk-neutral probability measures, but under the risk neutral probability measure the drift parameter is fixed by satisfying the martingale criterion.

There are two main categories of financial models. The first category are so-called *jump-diffusion* models. As the name suggests, the diffusion component must be present in such models. Under this category, the model stock price consists of two processes, the diffusion-process which determines the normal evolution of the price, and the fluctuation of price captured by a jump process. Here the jumps represent extreme events-stock market crashes and large drawdowns. Such an evolution can be represented by modelling log-prices as Lévy processes with a nonzero Gaussian component and a jump component, which is a compound Poisson process with finitely many jumps in every time intervals

$$Z_t = \theta t + \sigma W_t + \sum_{k=1}^{N_t} \xi_k \tag{2.67}$$

where $\{N_t, t \geq 0\}$ is the Poisson process counting the jumps of Z and ξ_k are i.i.d. random variables that represent jump sizes. The first model of this type is the *Merton* model[16] [50], which has been mentioned previously. This model suggests that the jumps in the log-price Z_t are assumed to be Gaussian distributed: $\xi_k \sim N(a, \delta^2)$. In the risk-neutral setting the characteristic exponent of the log stock price is represented as follows:

$$\psi(u) = -\frac{u^2\sigma^2}{2} + \lambda\left(e^{-\delta^2 u^2/2 + iau} - 1\right) - \text{iu}\left(\frac{\sigma^2}{2} + \lambda\left(e^{\delta^2/2+a} - 1\right)\right) \tag{2.68}$$

The second model of this type is the *Kou model* [42]. Kou (2002) assumed the structure in equation (2.67), but chose the jump distribution to be that of a two-sided exponential distribution. His choice of jump distribution was motivated by the fact that analysis of first passage time problems become analytically tractable which itself is important for the valuation of American put options. Under this model, the jump size ξ_k is a mixture of exponential distribution, i.e. in the form

$$\nu_0(dx) = (c_1\lambda_+ e^{-\lambda_+ x}\mathbb{I}_{x>0} + c_2\lambda_- e^{-\lambda_-|x|}\mathbb{I}_{x<0})dx \tag{2.69}$$

with $\lambda_+ > 0, \lambda_- > 0$ controlling the decay of the tail distribution of the positive and negative jump sizes and $p \in [0,1]$ representing the probability of an upward jump. The probability distribution of returns in this model has semi-heavy (exponential) tails.

The jump sizes under the described jump diffusion types of models have known distribution, the dynamical structure of the process are relative simple. One can easily simulate and apply Monte Carlo methods in pricing financial derivatives. However, the densities under such models have no close-form expression: moments and quantiles are quite hard to compute.

The second category of Geometric Lévy based models are called infinite activity models. In these models, one need not introduce a diffusion component since the dynamics of jumps is rich enough to generate nontrivial small time behaviour and it is arguably realistic in describing the price process at various time scales. Moreover, one can construct this class via Brownian subordination, which yields additional tractability over the jump-diffusion models. [12; 32]
There are two sub-categories under infinite activity models: infinite activity of finite variation and infinite activity of infinite variation. Examples of infinite activity of finite variation models are the VG models. Examples of infinite activity of infinite variation models are NIG models, and *stable models*.

[16] Alternative names are Merton jump diffusions or Compound Gauss-Poisson model.

2.7.6.2 Construction of Geometric Lévy models

There are three approaches to define a parametric Lévy process with infinite jump intensity. The first approach is to obtain a Lévy process by subordinating a Brownian motion with an independent increasing Lévy process (we called such a process a subordinator). Here one can immediately obtain the characteristic function of the resulting process. However an explicit formula for the Lévy measure in not always obtainable. Hence to account for non-normality of returns, one can write the return process $L_t = Z(T(t))$ as a subordinated process, where the subordinator $T(t)$ is an increasing Lévy process with stationary and independent increments and Z is a Gaussian process with independent increment. One example of a subordinated Lévy process is a Compound Poisson Process, that is a random walk time changed by a Poisson process. Hence one can say a subordinated Lévy process is still a Lévy process. Some Lévy process can be specified through Brownian subordination. This is an example of a time changed Lévy process. Brownian subordination involves specifying Brownian motion with drift and time-changing the process by a subordinator. Such subordinators act like "internal clocks", as market information arrives at a random time, the Wiener process with constant drift and volatility is evaluated by a given stochastic process (or time changed by a given stochastic process), if this stochastic process is infinitely divisible, then by simply time changing a drifted Wiener process, we obtain a new Lévy process. This feature makes Brownian subordinated models appealing in pricing derivative securities. An appropriate subordinator is required to specify a Brownian subordinated model and such subordinator should have the following properties.

- It should be able to give a realistic view of the randomness in information arrival.
- it should be able to control skewness and excess kurtosis of the underlying asset since it has enough parameters to incorporate these features.
- The expected time of information arrival at a calendar time should be equal to t.

It is worth mentioning that the generalised hyperbolic distribution (GHD) is constructed by subordinating a Brownian motion with a GIG process. Variance Gamma and Normal Inverse Gaussian both belong to the class of Generalised Hyperbolic Distribution.

The VG process is obtained by time-changing a Brownian motion with a gamma subordinator and the characteristic exponent of the form:

$$\psi(u) = -\log\left(1 + \frac{u^2\sigma^2}{2}\nu - iu\theta\nu\right)\Big/\nu$$

The Lévy measure of the VG process is given by

$$\nu(dx) = (C_1\mathbb{I}(x<0)\exp(-C_3x) + C_2\mathbb{I}(x>0)\exp(-C_4x))|x|^{-1}dx$$

which we will discuss in detail in the Chapter 4. The NIG process is obtained by subordinating a Brownian motion with a GIG process and the characteristic exponent is of the form:

$$\frac{1}{\nu} - \frac{1}{\nu}\sqrt{1 + u^2\sigma^2\nu - 2iu\theta\nu}$$

The second approach is to specify the Lévy measure directly. Some main examples are tempered stable processes [41], CMGY model [13].

The third approach is to specify the probability density function directly, more precisely, to specify the density of increments of the process at given time scales Δ, by arbitrarily selecting an infinitely divisible distribution. One example is the Generalised hyperbolic processes [23]. In this approach the increments of the process at the same node can be easily simulated and the distribution parameters can be estimated in a dataset that is sampled to the same period Δ. However, the law of the increments at other nodes is hard to find. Also, given the infinitely divisible distribution, one may not be able to tell whether the corresponding Lévy process has a Gaussian component, finite or infinite jump intensity as the associated Lévy Khintchine representation is hard to find.

Chapter 3

VWAP Options

VWAP options were suggested to reduce market manipulation risk, and a typical problem naturally arises is the pricing of VWAP options. An investor who wish to buy or sell a VWAP contract wants to have an idea of the fair price. The Market maker for these contract wishes to know how much the contracts are worth and how to hedge them. This gives rise to the need for derivative products base on VWAP. These derivative products are often called VWAP options. There are very few published or working papers available on the pricing of VWAP options. There exist only two papers and one dissertation which discusses VWAP from an option pricing point of view. The first contribution comes from Stace [79; 80]. The author approximates the distribution of VWAP to lognormal, the first two moments of VWAP are found by solving a system of nineteen ODEs, and finally the options are priced under the classical PDE approach. We briefly sketch what has been done now. *First*, the pricing PDE for the VWAP options is derived. Throughout the analysis, stock price is assumed to evolve as a GBM, volume is a mean reverting diffusion process[1], hence the dynamics of the underlying are given by the following SDE

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW_1(t), \\ dU(t) &= a(U(t))dt + b(U(t))dW_2(t) \end{aligned}$$

where $a : \mathbb{R} \mapsto \mathbb{R}$ and $b : \mathbb{R} \mapsto \mathbb{R}^2$ are sufficient regular functions so that the SDE has a unique global solution. The VWAP is represented as

$$I(t) \begin{cases} \frac{Y(t)}{Z(t)}, & \text{if } t > 0 \\ S(0), & t = 0 \end{cases}$$

where

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} S(s)U(s)ds \quad \text{and} \\ Z(t) &= \int_{t_0}^{t} U(s)ds; \qquad \int_{t_0}^{t} U(s)ds \neq 0 \end{aligned}$$

[1]Two different volume models were attempted, the first model is a Brennan & Schwarz type model of the form $dU(t) = \alpha(U_{\text{mean}} - U(t))dt + \beta U(t)dW_2(t)$; the second model is a CIR model of the form $dU(t) = \alpha(U_{\text{mean}} - U(t))dt + \beta\sqrt{U(t)}dW_2(t)$

Then

$$dY(t) = S(t)U(t)dt, \quad \text{and}$$
$$dZ(t) = U(t)dt$$

Where $Y(t)$ and $Z(t)$ are increasing functions of time when $S(t) > 0$ and $U(t) > 0$.
The author starts by postulating that the price of the option could be described by a function of t, S, U, Y, and Z, denoted as $V(t, S, U, Y, Z)$.

Then apply multidimensional Itô formula, V satisfies

$$\begin{aligned} dV &= V_t dt + V_S dS + V_U dU + V_Y dY + V_Z dZ + \frac{1}{2}(\sigma S)^2 V_{SS} dt + \frac{1}{2}(b(U))^2 V_{UU} + \rho\sigma S(b(U))V_{SU} dt \\ &= V_S dS + V_U dU + \mathcal{L}V dt \end{aligned}$$

where $\mathcal{L} = V_t + USV_{YY} + UV_Z + \frac{1}{2}(\sigma S)^2 V_{SS} + \frac{1}{2}(b(U))^2 V_{UU} + \rho\sigma Sb(U)V_{SU}$.

Meanwhile, a portfolio composed of two options (long position of 1 unit and short position of Δ_1 unit) and one stock (long position of 1 unit) is constructed to arrive a PDE that describes the price of the option, i.e.

$$\Pi = V - \Delta S - \Delta^{(1)} V^{(1)}$$

Again, apply Itô formula,

$$\begin{aligned} d\Pi &= dV - \Delta dS - \Delta^{(1)} dV^{(1)} \\ &= V_S dS + V_U dU + \mathcal{L}V dt - \Delta dS - \Delta^{(1)}(V_S^{(1)} dS + V_U^{(1)} dU + \mathcal{L}V^{(1)}) \end{aligned}$$

Assume that $V_U \neq 0$, $V_U^{(1)} \neq 0$ and choose Δ and $\Delta^{(1)}$ such that the stochastic term dS and dU vanish, the change in the portfolio is given by

$$d\Pi = (\mathcal{L}V - \Delta^{(1)}\mathcal{L}V^{(1)})dt$$

Which is a difference of two continuous local martingale and has finite variation. After some steps, the fundamental PDE he used in describing the price of the VWAP option is arrived as

$$\begin{aligned} V_t + \frac{1}{2}(\sigma S)^2 V_{SS} + \frac{1}{2} b^2(U) U_{SS} + \rho\sigma Sb(U)V_{SU} \\ + rSV_S + SUV_Y + UV_Z - rV + (a(U) - b(U)\Lambda(t, S, U, Y, Z))V_U = 0 \quad (3.1) \end{aligned}$$

where $\Lambda(t, S, U, Y, Z)$ is some arbitrary constant function. Knowing that the Feynman-Kac formula provides a convenient link between PDE and expectation, the expectation of the price of VWAP is found. *Second*, the author derives some analytical formulae for the bounds for the option that are independent of the volume process. He emphasises these bounds could be viewed as a hedging strategy that covers all risk, however, the cost is prohibitively big which makes the VWAP unattractive to buy. *Third*, Monte Carlo analysis is developed to price and find the Greeks of the VWAP options, several control variates are found. *Fourth*, the author postulates that the VWAP moments represent effective lognormal moment and the price of the option is found by matching the first two moments of the VWAP to a lognormal distribution. To approximate the expectation and variance of the quotient $\frac{Y}{Z}$, the following Taylor series expansion is used [57]

$$\mathbb{E}\left(\frac{Y}{Z}\right) \approx \frac{\mathbb{E}(Y)}{\mathbb{E}(Z)} - \frac{\mathrm{Cov}(Y,Z)}{(\mathbb{E}(Z))^2} + \frac{\mathbb{E}(Y)}{(\mathbb{E}(Z))^3}\mathrm{Var}(Z) \ ,$$
$$\mathrm{Var}\left(\frac{Y}{Z}\right) \approx \left(\frac{\mathbb{E}(Y)}{\mathbb{E}(Z)}\right)^2 \left(\frac{\mathrm{Var}(Y)}{(\mathbb{E}(Y))^2} + \frac{\mathrm{Var}(Z)}{(\mathbb{E}(Z))^2} - 2\frac{\mathrm{Cov}(Y,Z)}{\mathbb{E}(Y)\mathbb{E}(Z)}\right) \ .$$

The first two moments are found by solving a large (19 equations) system of ODEs. Subsequently, the lognormal parameters $\tilde{\mu}$ and $\tilde{\sigma}$ are found and the fundamental pricing PDE that describes the price of VWAP options is solved[2]. In addition to price vanilla type of VWAP call option, the author also attempts to price an exotic type of VWAP option: a VWAP digital option. *Fifth*, a finite difference method[3] is attempted to solve the PDE of the form (3.1) , in particular, explicit (See [84]), Crank-Nicolson [16], and Alternating Direction Implicit (ADI) scheme [65] are used. Solving by finite difference is found to be very challenging as there are four state variables[4] together with the time variable, and the boundary condition is quite hard to formulate. As a *final* step, a series solution to the price of the VWAP options is developed and the first two term of this solution are explicitly derived, based on the assumption that stock price follows geometric Brownian motion and the volume is a mean reverting process. It is found that the first term of the series described by a PDE which is quite similar to the Asian option PDE. The first term is found to be independent of the volume process. The author has also shown that the first term of the expansion works quite well when mean reversion is high.

The second approach is due to Novikov et al.(2010) [62], the authors propose to price the VWAP option by matching moments. However, the problem was solved via a semi-analytical approach. Now let's sketch briefly. First, the authors postulate that the VWAP moments represent a particular class of Lévy process, so-called Generalised Inverse Gaussian (GIG) process. The rationale is that the VWAP option is very similar to the Asian arithmetic option. There is a vast literature on the pricing of Asian arithmetic options. In the case of Asian arithmetic options, some authors suggest moment-match to skewed distribution, i.e. Inverse Gamma (See[51]). A skewed distribution belongs to the class of infinitely divisible distribution and so induce a Lévy process $X = \{X(t), t \geq 0\}$. Therefore, it is not unreasonable to postulate the process of VWAP represents an effective GIG process. Another important observation is that, the expression of VWAP is in term of the following ratio of two integrals,

$$\mathbb{E}\left(\frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt}\right) \tag{3.2}$$

then taking the expectation, they assume S_t is assumed to be independent[5] to U_t, hence equation (3.2) becomes

$$\int_0^T \mathbb{E}S_t \mathbb{E}\left(\frac{U_t}{\int_0^T U_t dt}\right)$$

Knowing the joint Laplace Transform

$$\Phi(z, r, q) = \mathbb{E}(\exp\{-zU_t - rU_s - qV_T\})$$

[2]See section 6 of [80] or p153 of [79].

[3]The finite difference method is frequently employed to solve the PDEs which describe the option prices. The method is well-established, and there is a vast literature about the topic. Some of the excellent references are Morton& Mayer (2005) [59], Strikwerda (1989) [81], Fletcher (1991) [28] and Mitchell & Griffiths (1980) [53]

[4]Random variables that depend on the time. The number of state variables determine the number of dimensions in a PDE.

[5]As discussed in Chapter 1, the independence between the Brownian motion under the price dynamics and the Brownian motion under the volume dynamics leads to the independence between S_t and U_t.

and assuming the usual technical condition holds ,i.e.

$$(U_t/V_T) < \infty$$

It follows that

$$\begin{aligned}
\mathbb{E}\left(\frac{U_t}{V_T}\right) &= -\int_0^\infty \frac{\partial}{\partial z}\Phi(z,0,q)\bigg|_{z=0} dq \\
&= \int_0^\infty \mathbb{E}(U_t e^{-qV_T})dq \\
\mathbb{E}(A_T) &= -\int_0^T \mathbb{E}(S_t)\left(\frac{\partial}{\partial z}\Phi(z,0,q)\bigg|_{z=0} dq\right)dt
\end{aligned}$$

Clearly, to compute the first moment, the idea is to find $\Phi(z,0,q)$. However, to compute the second moment, $\Phi(z,r,q)$s need to be derived. This is the joint Laplace Transform of U_t, U_s and the integral of U_t. The key is to compute the Laplace Transform of the integral of the squared Ornstein-Uhlenbeck process, which does not involve solving any ODEs or PDEs, but rather utilising the Girsanov theorem. Having analytical moments derived, VWAP moments are matched to the GIG distribution; GIG parameters are found and are used to specify the distribution for the VWAP; Consequently Option prices are found by integrating the terminal payoff against the state price density.
The structure of this chapter is as follows. First, the VWAP is described in detail. Then the specific contract priced in this thesis is described. Finally the use of a VWAP option is presented.

3.1 The Volume Weighted Average Price

A plain vanilla Asian option depends on the time weighted (or arithmetic average) stock price. The weight on the stock price is equally given to days of light trading and days of heavy trading. A VWAP is quite similar to the arithmetic average price, except it differs on how the average is taken. In practice, trade count is required to calculate the VWAP, this explains why VWAP is often used for computing the stock prices for companies that are publicly listed. The VWAP assigns more weight to high trading periods than thin trading periods. Therefore, one main reason that VWAP is introduced is that it helps reducing market manipulation risk. As a result, the use of VWAP is more appropriate for underlyings that are public listed securities. Nevertheless, we give a simple example to see the difference of a volume weighted average and an arithmetic average.

Example 3.1 (VWAP and Arithmetic Comparison)**.** Suppose a stock trades at \$5 today and there are 10 trades; tomorrow it trades at \$50 and there is 1 trade. The volume weighted average price $\frac{\$5\times10+\$50\times1}{10+1} = \$9.09$, while arithmetic average price is $\frac{\$5+\$50}{2} = \$27.5$

Now we define VWAP in both continuous and discrete time. We start with the more practical discrete case.

Definition 3.1 (Discrete-time VWAP)**.** Let the time interval $[t_0, t]$ be discretized into the intervals $t_0 = \hat{t}_0, \hat{t}_1, ..., \hat{t}_N = t$ and the volume weighted average is formed as

$$A(T) = \sum_{i=1}^{N} w_i S(t_i) \quad \sum_{i=1}^{N} w_i = 1, \quad w_i = \frac{U(\hat{t}_i)}{\sum_{i=1}^{N} U(t_i)}$$

where N represents the number of transactions, $S_i = S(\hat{t}_i)$, $U_i = U(\hat{t}_i)$ denote the price and volume for each i-th transaction, respectively, $i = \{1, ..., N\}$, $A(t_0) = S(t_0)$ and $\sum_{i=0}^{N} U(\hat{t}_i) \neq 0$. Alternatively, VWAP can be defined in continuous time.

Definition 3.2 (Continuous-time VWAP)**.** Denote the price of a stock at time t as $S(t)$, and the number of trades of $S(t)$ per unit time as $U(t)$. Let $t_0 < t$, thus the total value value of admissible trades during the interval $[t_0, t]$ is $\int_{t_0}^{t} S(u)U(u)du$, and the number of shared traded is $\int_{t_0}^{t} U(u)du$. The continuous VWAP over time interval $[t_0, t]$ is defined as

$$A(t) = \begin{cases} \frac{\int_{t_0}^{t} S(u)U(u)du}{\int_{t_0}^{t} U(u)du}, & t_0 < t \\ S(t), & t = t_0 \end{cases}$$

Assume that $\int_{t_0}^{t} U(s)ds \neq 0$.

Remark 3.1. Definition 3.2 is reduced to the definition of an Asian arithmetic option when $U(t)$ is constant.

It is instructive to visualise how VWAP behaves in practice via the plot of real data. We deliberately omit here due to the lack of access to real-time database. However, it worth mentioning that Stace (2006) [79] claims that there is little difference between an arithmetic average price and volume weighted average price (VWAP). The author starts with a collection of real intra-day data during one month period for four public-listed stock and constructs running Arithmetic and volume weighted average price. Then by plotting the two averaging prices against the real-time stock price and it is found the only difference is that VWAP tends to fluctuate more around the start of the averaging before settling down. During the observation the author also identifies one situation where the arithmetic average differs from the volume weighted average quite a lot. This is the situation when a stock price drops to zero and stays there, or stops trading, in the case of the running arithmetic, the average tends to decrease for the remaining time when averaging is re-computed. On the other hand, the VWAP tends to be invariant. Hence it is quite clear that the two different ways of averaging potentially impact on the derivative payout.

3.2 VWAP Options

This thesis is concerned with the valuation of European call VWAP options. We consider fixed strike with a payoff at the terminal time of

$$C(T) = (A(T) - K)^{+} = \left(\frac{\int_0^T S(t)U(t)dt}{\int_0^T U(t)dt} - K \right)^{+}$$

Once the call options contracts are priced, put options contracts can be also price via the use of put-call parity.

3.3 The use of a VWAP

The volume weighted average price (VWAP) over rolling number of days in the averaging period is used as a benchmark price by market participants and can be regarded as an estimate for the price that a passive trader will pay to purchase securities in a market. During the past few years, as institutional investor try to get a real understanding of the true cost of implementing a managerial or strategy change, VWAP has been gaining in popularity to measure equity execution. VWAP represents the average price of a security weighted by trade volume. In other words, It is a simple way to calculate the average price of a stock over any given time periods. Nowadays, it is commonly used in brokerage houses as a quantitative trading tool and also appears in Australian taxation law to specify the price of share-buybacks[6]

[6] In Australian financial market, share-buyback is a common activity among Investment Banks, Insurance companies and other large scale financial institutions in gaining tax advantage and reducing managerial agency problem.

Chapter 4

The Variance Gamma process and model

As discussed previously (2.7.6.1), models with jumps usually fall into two categories: Jump-diffusion models and infinite activity (intensity) models. In particular, the jump size under the first category can be chosen either to be Gaussian or non-Gaussian type; Likewise, The infinite activity models can either be chosen to be of finite variation or infinite variation. A drawback of the jump-diffusion models is the parameter instability due to the infinite variation property possessed by the Brownian component. In this regard, Madan and Seneta propose a pure jump model (VG model) in modelling the dynamics of the stock price. This model is a infinite activity based model with finite variation. The nice properties of the VG model have led to its recent implementation in the Bloomberg system through the function SKEW [38]. Now let's take a closer look at VG.

4.1 The VG process and distribution

The class of Variance Gamma distribution was first introduced by Madan & Seneta (1990) [48] in the late 1980s. The symmetric case of the VG process was proposed and developed by Madan & Seneta [48] and Madan & Miline [52] as a model for studying stock returns and option pricing. The original formulation was further generalised by Madan, Carr and Chang (1998) [47] to a general VG model. The original symmetric VG process is considered as a special case of the general case with $\theta = 0$. We always refer to the general case whenever we talk about the VG process in this thesis. The VG process has become one of the most popular Lévy models among academia and practitioners.

The VG process is a process of independent and stationary increments, that is a Lévy process [31]. A Lévy process can be represented as the sum of three independent components: a deterministic drift, a continuous Wiener process, and a pure jump process. Brownian motion is a special case where the jump component is absent. On the other hand, Poisson process is a special case where the Brownian motion and the deterministic component are absent. VG process is a pure jump process, which is similar to the Poisson process, and thus it can be expressed in terms of its Lévy density, the simplest version with no parameters being (See [48])

$$k_{VG}(x) = \frac{1}{|x|} e^{-\sqrt{2}|x|} \tag{4.1}$$

For the VG process with the usual (θ, σ, ν) parameterisation, the Lévy density is given by

$$k_{VG}(x) = \frac{1}{\nu|x|} \exp\left(\frac{\theta}{\sigma^2} x - \frac{1}{\sigma} \sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}} |x| \right)$$

where $\nu, \sigma > 0$

The Lévy measure has the following form

$$\nu(dx) = (C_1 \mathbb{I}(x < 0) \exp(-C_3 x) + C_2 \mathbb{I}(x > 0) \exp(-C_4 x))|x|^{-1} dx$$

where $C_1 > 0, C_2 > 0, C_3 > 0, C_4 > 0$ are constants. Carr-Geman-Madan-Yor have introduced CGMY process in [12], which is an extended process of VG process. The Lévy measure of the CGMY process is

$$\nu(dx) = C(\mathbb{I}(x < 0) \exp(-Gx) + \mathbb{I}(x > 0) \exp(-Mx))|x|^{-(1+Y)} dx \tag{4.2}$$

where $C > 0, G > 0, M \geq 0, Y < 2.$ and $Y \leq 0$, then $G > 0$ and $M > 0$ are assumed. When $Y = 0$, we obtain a VG model as a difference of two independent identically distributed gamma processes.

Where

$$C = 1/\nu > 0$$

$$G = 1 \Bigg/ \left(\sqrt{\frac{\theta^2 \nu^2}{4} + \frac{\sigma^2 \nu}{2}} - \frac{\theta \nu}{2} \right)$$

$$M = 1 \Bigg/ \left(\sqrt{\frac{\theta^2 \nu^2}{4} + \frac{\sigma^2 \nu}{2}} + \frac{\theta \nu}{2} \right)$$

Unlike the Poisson process, the VG process may have infinite number of jumps in any interval, making it a process of infinite activity. Unlike Brownian motion, the VG process has finite variation, so in some sense it behaves in a more stable way.

4.1.1 The Construction of a VG process

There are two representations for the VG process, both of which are useful, but in different context. In the first representation, the VG process is interpreted as a Brownian motion with drift, where time is changed by a gamma process. Suppose there is a Wiener process with constant drift θ and volatility σ. If $W(t)$ is the standard Brownian motion, we can write the process $B(t; \theta, \sigma)$ as

$$B(t; \theta, \sigma) = \theta t + \sigma W(t) \tag{4.3}$$

where the time t follows a gamma process $\tau_\gamma(t) := \gamma_t^{(\nu)} \sim \Gamma(\frac{t}{\nu}, \nu)$, i.e.

$$f_\gamma(x, t) = \frac{x^{\frac{t}{\nu} - 1} e^{-\frac{x}{\nu}}}{\nu^{\frac{t}{\nu}} \Gamma\left(\frac{t}{\nu}\right)}$$

with unit drift $\mu = 1$ (mean rate per unit of time) and variance parameter ν which results in the pure jump process that has an infinite number of jumps in any interval of time :

$$\begin{aligned} X(t;\theta,\sigma,\nu) &= B(\tau_\gamma(t),\theta,\sigma) \\ &= \theta\tau_\gamma(t) + \sigma W_{\tau_\gamma(t)} \end{aligned} \tag{4.4}$$

The gamma density has the characteristic function, $\varphi_\gamma(u,t) = \mathbb{E}(\exp(iu\gamma_t^{(\nu)}))$, given by

$$\varphi_\gamma(u,t) = \mathbb{E}(\exp(iuX_t^\gamma)) = \left(\frac{1}{1-iu\nu}\right)^{\frac{t}{\nu}} \tag{4.5}$$

The characteristic function for the VG is obtained quite easily by first conditioning on the gamma process and then using the gamma characteristic function to get

$$\varphi_{VG}(u,t) = \mathbb{E}(\exp(iuX_t^\gamma)) = \left(\frac{1}{1-iu\theta\nu+\frac{\sigma^2\nu}{2}u^2}\right)^{\frac{t}{\nu}} = e^{t\psi(u)} = \exp\left\{(\ln\left(\frac{1}{1-iu\theta\nu+\frac{u^2\sigma^2\nu}{2}})\right)^t\right\} = (f(u))^t \tag{4.6}$$

The characteristic function of the form (4.6) is in the class of infinitely divisible distributions, hence one can deduce the Lévy measure $k_{VG}(x)$ from the logarithm of the characteristic function:

$$\log(\varphi_{VG}(u,t)) = t\int_{-\infty}^{\infty}(e^{iux}-1)k_{VG}(x)dx \tag{4.7}$$

Differentiating equation (4.7) with respect to u on both sides yields a recognisable Fourier transform, leading to

$$k_{VG}(x) = \frac{c}{|x|}e^{-A|x|+Bx}, \tag{4.8}$$

$$A = \frac{1}{\sigma_{VG}}\sqrt{\left(\frac{2}{\nu}+\frac{\theta^2}{\sigma_{VG}^2}\right)}, \tag{4.9}$$

$$B = \frac{\theta}{\sigma_{VG}^2}, \tag{4.10}$$

$$c = \frac{1}{\nu} \tag{4.11}$$

where $B < A$ and $|x|k_{VG}(x)$ is a decreasing exponential for positive x and increasing exponential for negative x. This interesting property of the Lévy measure is a necessary and sufficient condition for the unit time random variable to be self-decomposable [74]. Let γ^ν be a gamma process with unit drift and variance ν. One can describe the dynamics of the continuous time gamma process by describing the simulation of the process. Since the gamma process is an infinitely divisible distribution of i.i.d. increments over non-overlapping intervals of equal length, the simulation may be described in terms of the Lévy measure [71].

$$\nu_\gamma(dx) = \frac{\exp\left(-\frac{1}{\nu}x\right)}{\nu x}dx, \quad \text{for } x > 0 \text{ and } 0 \text{ otherwise.} \tag{4.12}$$

Another interesting remark is that, the Lévy measure has an infinite integral, one can see that the gamma process has an infinite arrival rate of small jumps, as is indicated by the concentration of the Lévy measure at the origin. The process is of pure jump type and could be approximated as a compound Poisson process. To simulate the compound Poisson process approximation, we truncate the Lévy measure near the origin by throwing away small jumps of size below ϵ. Then the area under the

truncated Lévy measure can be used as the Poisson arrival rate of jumps. The normalised truncated Lévy measure acts as the conditional density of jump magnitudes, given the arrival of a jump.
The VG process $X(t;\theta,\sigma,\nu)$ can now be formally defined in terms of the Brownian motion with drift $B(t;\theta,\sigma)$ and gamma process with unit mean rate $\tau_\gamma(t)$ as

$$X(t;\theta,\sigma,\nu) = B(\tau_\gamma(t),\theta,\sigma) \tag{4.13}$$

The idea of time change makes economic sense. It is known that the financial market does not evolve identically every day. More precisely, trade volume is not uniform during the day and trading activities fluctuate quite a lot form time to time. Intuitively, the original clock could be regarded as the calendar time and the random clock could be regarded as the "business time". This business clock could be tuned faster while trading activity is high during a business day, and vice versa, when there is not much trading it could be slowed down. Hence, conceptually, the business time can be distinguished from calendar time and describe the evolution of trading activity. One can view the VG process as a Brownian motion run under a random gamma clock.

From the Lévy measure, one may infer that the VG process is also the difference of two independent increasing gamma processes. That is

$$X_t^{VG} = \gamma_t^{(\mu_+,\nu_+)} - \gamma_t^{(\mu_-,\nu_-)} \tag{4.14}$$

Where $\mu_\pm$ are defined in equation (5.5). Here the two gamma processes are independent (but defined on a common probability space) with parameters

$$\mu_\pm = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} \pm \frac{\theta\nu}{2} \tag{4.15}$$

$$\nu_\pm = \mu_\pm^2\nu \tag{4.16}$$

This representation allows us to determine the Lévy measure for X_t,

$$\nu(dx) = \begin{cases} \frac{{\mu^+}^2}{\nu^+} e^{-\frac{\mu^+}{\nu^+}|x|}/|x|, & \text{if } x > 0 \\ \frac{{\mu^-}^2}{\nu^-} e^{-\frac{\mu^-}{\nu^-}|x|}/|x|, & \text{if } x < 0 \end{cases} \tag{4.17}$$

The division by the absolute value of the jump size in the VG Lévy measure (4.17) explains why the VG process has infinite activity, as the VG Lévy measure integrates to infinity. It is also clear that the process is of finite variation as $|x|$ is integrable with respect to the VG Lévy density. In term of the generating triplet (σ,ν,θ), the Lévy measure can be re-written as

$$\nu_{\text{VG}}(dx) = \frac{\exp(\theta x/\sigma^2)}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma^2}|x|\right) dx \tag{4.18}$$

Now we compare VG process against Brownian motion, recall the definition of a standard Brownian motion $W = \{W_t, t \geq 0\}$

- The trajectories of W are $\mathbb{P}$-a.s. continuous.
- It starts at zero: $W(0) = 0$ or $\mathbb{P}(W(0) = 0) = 1$.
- Stationary increment: $\forall\ 0 \leq s \leq t,\ W(t) - W(s) \stackrel{d}{=} W(t+h) - W(s+h)$.
- Distribution identity (Normal Increment): $\forall\ 0 \leq s \leq t,\ W(t) - W(s) \stackrel{d}{=} W(t-s)$.

- Independent increment: $\forall \;\; 0 \le u \le s \le t$, $W(t) - W(s)$ is independent of $\{W(u), u \le s\}$

- $W(t+h) - W(t) \sim N(0,h)$: increments are normally distributed.

In a similar manner, one can define the stochastic process based on the VG distribution. (See [76] for example). A stochastic process $X = X(t), t \ge 0$ is a Variance Gamma process with parameter C_1, C_2, C_3, C_4 if

- The trajectories of X are $\mathbb{P}$-a.s. right continuous with left limits.

- It starts at zero: $X(0) = 0$ or $\mathbb{P}(X(0) = 0) = 1$.

- Stationary increment:$\forall \;\; 0 \le s \le t, \;\; X(t) - X(s) \overset{d}{=} X(t+h) - X(s+h)$.

- Distribution identity:$\forall \;\; 0 \le s \le t, \;\; X(t) - X(s) \overset{d}{=} X(t-s)$.

- Independent increment: $\forall \;\; 0 \le u \le s \le t$, $X(t) - X(s)$ is independent of $\{X(u), u \le s\}$

- $X(t+h) - X(t) \sim$ VG $(C_1(h), C_2(h), C_3, C_4)$: increments are VG distributed.[1]

It turns out that a VG process is a pure jump process. Sample paths have no diffusion component in contrast with Brownian motion.

4.2 The VG Stock Price Model

The VG model is a reasonably tractable and parsimonious model among all pure jump models. The Madan et al. [47] paper shows that the VG process is successful in explaining the volatility smile due to the fact that VG process is a purely discontinuous process. Comparing to the GBM model which contains two components: deterministic drift and diffusion components, the Geometric Lévy based stock price model we consider in this thesis contains three components : deterministic component, diffusion & jump component. By modelling the log-returns of the stock price with a general Lévy process (a combination of diffusion process and the VG process),

$$L_t = \log(S_t) - \log(S_{t-1}) = m + \sigma_{BM}(W_t - W_{t-1}) + (X_t^{VG} - X_{t-1}^{VG})$$

and the stock price process can be written as

$$S_t = S_0 e^{L_t}$$

the VG model can capture the well-documented volatility smile/skew observation. Assuming no dividends are paid, a risk-neutral measure is chosen to mean-correct the original. We adopt the difference-of-gammas representation and define the continuous stock price model as

$$S_t = S_0 e^{L_t}, \quad S_0 > 0 \tag{4.19}$$

where $L_t = mt + \sigma_{BM} W_t + X_t^{VG}$, assuming 0 interest rate.

In this way, the log-returns of stock prices are no longer normally distributed.
Now we recall some basic facts under the classical Black Scholes framework,

[1]When $C_1 = C_2$, we obtain the CGMY process, then $X(t+h) - X(t) \sim (C(h), G, M)$.

$$\log S_{t+1} - \log S_t \sim \text{Gaussian}\left(\mu_{BM} - \frac{\sigma_{BM}^2}{2}, \sigma_{BM}^2\right)$$

One can move easily from the real world to the risk-neutral world by simply replacing the drift μ with a constant interest rate r (assume no-dividends).

$$S_t = S_0 \exp\left((r - \frac{\sigma_{BM}^2}{2})t + \sigma W_t\right),\ t \geq 0$$

In contrast with the Black Scholes' setting; in the case of a Geometric Lévy model, we are working in an incomplete market, meaning that there is no unique transformation. In fact, there are infinite many E.M.M.. However, knowing market doesn't have to be complete for the absence of arbitrage to occur, one particular simple transformation is the mean-correcting measure changes, where the Lévy process is shifted in such a way as to obtain a martingale.

$$S_t = S_0 e^{L_t} = S_0 e^{mt + \sigma_{BM} W_t + X_t^{VG}}$$

where

$$m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log\left(\frac{1}{1 - \theta v - \frac{1}{2}\sigma_{VG}^2 \nu}\right) \Big/ \nu \tag{4.20}$$

ensures $e^{-rt}S_t$ is a martingale.
where the jump Lévy process X_t^{VG} is the difference of 2 independent gamma processes $X_\gamma(C_1; 1/C_4), X_\gamma(C_2; 1/C_3)$.That is

$$X_t^{\mathrm{VG}} = X_\gamma(C_1; \frac{1}{C_4}) - X_\gamma(C_2; \frac{1}{C_3}) \tag{4.21}$$

where the constant m is chosen in such a way that the discounted asset price is a martingale; that is, it must satisfy

$$\mathbb{E}(e^{-rt}S_t) = S_0$$

Hence, the drift

$$m = \mu - \frac{1}{2}\sigma_{\mathrm{BM}}^2 - \log\left(\frac{1}{1 - \theta v - \frac{1}{2}\sigma_{\mathrm{VG}}^2 \nu}\right) \Big/ \nu$$

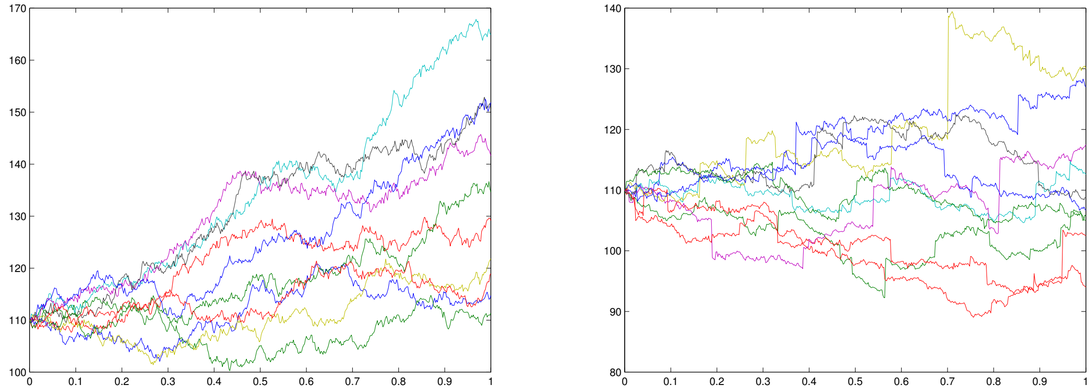
is obtained.

Proof. Using Lévy Khintchine theorem and the basic property of martingales

$$\begin{aligned} \mathbb{E}S_t &= S_0 e^{t\psi(1)} = S_0 \\ \Longrightarrow \mathbb{E}S_t &= S_0 e^{t\psi(1)} = S_0 e^{t\left(m + \frac{\sigma_{\mathrm{BM}}^2}{2} + \log\left(\frac{1}{1 - \theta\nu + \frac{(\sigma_{\mathrm{VG}})^2}{2}\nu}\right) \Big/ \nu\right)} = S_0 \end{aligned}$$

iff

$$m = -\frac{1}{2}\sigma_{\mathrm{BM}}^2 - \log\left(\frac{1}{1 - \theta\nu + \frac{\sigma_{\mathrm{VG}}^2}{2}\nu}\right) \Big/ \nu \quad \text{assuming no growth in price}$$



(a) Trajectories of GBM stock path

(b) Trajectories of Geometric Lévy process with VG jumps

Figure 4.1: Comparison of trajectories

$$m = \mu - \frac{1}{2}\sigma_{\text{BM}}^2 - \log\left(\frac{1}{1 - \theta\nu + \frac{\sigma_{\text{VG}}^2}{2}\nu}\right)\Big/\nu \quad \text{assuming price growths at rate } \mu$$

□

There are two structural properties that make the VG model appealing. The first one is the complete monotonicity of the jump size intensity, meaning that the large jumps occur less frequently than small jumps. This is practically sensible as more buy orders are placed, price naturally increases, and as more sell orders are placed, price naturally decreases. Market participants from both buy and sell sides have incentives to minimise such impacts. The second structural property is the infinite activity property, in the sense preserves the sample path property of Brownian motion. It is instructive to visualise the difference of a geometric Lévy process and a lognormal process (or GBM). See Fig.4.1. The left are ten trajectories of the well-known Geometric Brownian Motion, while on the right are ten trajectories of geometric Lévy process with variance gamma jumps. As one may tell that, on the right plot, there are large number of small jumps. And the magnitude of the jumps progressively concentrates on the origin. In this sense, the VG model respects the intuition underlying the sample path continuity of Brownian Motion as a model.

Chapter 5

Pricing VWAP Options via moment-matching

The payoff of a VWAP option depends on the following path-dependent random variable

$$A_T = \frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt}$$

In this chapter, we study an approximation method to price the VWAP options. The stock price is assumed to evolve as a geometric Lévy process and the trade volume follows a squared Ornstein-Uhlenbeck process. However, due to the complexity of the trade volume process, the moments of VWAP process are hard to calculate without further approximation. Hence, we approximate via a technique so-called moment matching, which is a common method used in pricing Asian options. Since the moments of the lognormal law are well known, the moments of VWAP were initially matched to the lognormal distribution and the lognormal parameters are subsequently found. Following the idea in Novikov et al. (2010), we then attempt to match to the Generalise Inverse Gaussian (GIG) distribution, which is a Lévy process with three parameters. The rationale of our choice of approximating to a GIG distribution is the following.

First, we recognise that the VWAP options are quite similar to the Asian arithmetic options in the case when volume $U(t)$ is considered to be constant. It is well-known that the Asian arithmetic options are quite difficult to price and hedge as they do not possess close-form analytic solutions in the classical Black Scholes framework. The main reason for this difficulty is that the payoff depends on a finite sum of correlated lognormal variables, which is not lognormal and a suitable probability distribution cannot be identified [51]. In the case of VWAP option, this difficulty is compounded by the complexity of the volume process. This is because the VWAP pricing results strongly depend on volume type and it is known that the VWAP process cannot be reconstructed from market prices [11]. Currently there is no consensus in the literature on the process that could describe the dynamics of the volume accurately. However, with the available techniques on the pricing of Asian options, we may be able to develop an analogous method to deal with this challenge. The Milevsky and Posner (1998) [51] paper postulates that the infinite sum of the lognormal random variables tend to be Inverse Gamma distributed[1], under some certain restrictions on the parameters. Based on the postulation this result is used to approximate the finite sum of the correlated random variables and a closed form expression for the value of the Asian arithmetic option is found by using the Inverse Gamma density as the state-price density[2]. The idea

[1] This means the reciprocal of the random variable is gamma distributed.

[2] A state price is price of an Arrow Debreu Security, it is the value today of 1 dollar paid in one state of the world

from the Milevsky and Posner (1998) paper is explored and applied to the pricing of VWAP option in Novikov et al. (2010). The authors postulate that the terminal VWAP price could approximate to a GIG distribution, the GIG distribution is infinitely divisible and so it is a Lévy process. As mentioned previously, there is a vast literature on the pricing of options under Lévy processes, a popular class of Lévy process that is often used in exotic option pricing is the Normal Inverse Gaussian (NIG) process [2]. As we have seen in chapter 2, the NIG process is a special case of the GIG when the index of the Bessel function is taken as $-\frac{1}{2}$. In addition, the extra parameter of the GIG distribution might make it a more flexible distribution to model stock prices. Furthermore, The relationship between the quantity similar to VWAP and some skewed distribution (gamma, NIG, GIG) has been extensively studied in the actuarial literature (Dufresne, Gerber and Shiu (1991) [22] and Chaubey, Garrido and Trudeau (1998) [14]). Hence instead of matching the well known lognormal, we have chosen to approximate the distribution of the VWAP by a GIG process.

5.1 The Model

First, S_t, U_t processes are defined. Let $W_t = (\widehat{W_t}, \widetilde{W_t}), 0 \leq T$ be two dimensional Brownian Motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Assume

$$S_t = S_0 e^{rt+mt+X_t^{VG}+\sigma_{BM}(\rho,\sqrt{1-\rho^2})\cdot W_t}$$

where $\sigma_{BM} \in \mathbb{R}^+$. The dynamics of the Ornstein-Uhlenbeck process are described as

$$dX_t = \lambda(a - X_t)dt + (\sigma_{OU}, 0) \cdot dW_t$$

for some real σ_{OU}. Let introduce the real-valued stochastic processes $\bar{W}^{(1)}$ and $\bar{W}^{(2)}$ by setting

$$\bar{W}^{(1)} = \widehat{W_t},\ \bar{W}_t^{(2)} = \rho \widehat{W_t} + \sqrt{1-\rho^2}\widetilde{W_t}$$

where $\rho \in [-1, 1]$, $\bar{W}_t^{(1)}$ and $\bar{W}_t^{(2)}$ are standard one-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with quadratic variation satisfying $[\bar{W}^{(1)}, \bar{W}^{(2)}]_t = \rho t$. It is evident that

$$S_t = S_0 e^{rt+mt+X_t^{VG}+\sigma_{BM}\bar{W}_t^{(2)}} \tag{5.1}$$

and

$$dX_t = \lambda(a - X_t)dt + \sigma_{OU} d\bar{W}_t^{(1)} \tag{5.2}$$

where $\bar{W}_t^{(2)}$ is a Brownian motion correlated with the Brownian motion $\bar{W}_t^{(1)}$ for the volume process U_t, X_t^{VG} is a variance gamma process independent of $\bar{W}_t^{(2)}$, and $m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log(1 - \theta\nu - \frac{1}{2}\sigma_{VG}^2\nu)/\nu$ is a drift to ensure $e^{-rt}S_t$ is a martingale.

Stace in his paper [80] used mean-reverting processes (CIR and Brennan-Schwartz processes) for modelling trade volume. Here we adopt the following mean-reverting process (shifted squared Ornstein-Uhlenbeck process) to model the trade volume in accordance to Novikov et al. (2010),

$$U_t = X_t^2 + \delta, \quad dX_t = \lambda(a - X_t)dt + \sigma_{OU} d\bar{W}_t^{(1)}, \quad X_0 = a$$

where $m, \lambda, \sigma_{\text{BM}}, a, \delta$ are assumed to be bounded constants, $\delta \geq 0, \lambda > 0$. In particular, λ being the speed of mean reversion, a can be thought of as the long term average of the volume process. Intuitively,

(only) tomorrow.

when X_t rises, the drift < 0, then X_t tends to drop ; when X_t drops, the drift > 0, then X_t tends to rise back again. The Ornstein-Uhblenbeck process X_t is represented as

$$X_t = a + \sigma_{OU}\xi_t \tag{5.3}$$

where ξ_t is a standard Ornstein-Uhblenbeck process satisfying the SDE

$$d\xi_t = -\lambda\xi_t dt + d\bar{W}_t^{(1)}, \xi_0 = 0 \tag{5.4}$$

In the symmetric case, when $\delta = 0$ and $a = 0$, the process $U(t)$ is a particular case of the Cox-Ingersoll-Ross(CIR) process, [60].
S_t is assumed to depends on U_t for any $t \geq 0$.
The continuous time analog of the VWAP is given by

$$A_T = \frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt}$$

The approximation of $\mathbb{E}(A_T)$ and $\mathbb{E}(A_T^2)$ are required, which we will discuss shortly.

5.2 The Approximation

A common approximation to the Asian arithmetic average option is to approximate the distribution of the underlying asset by a lognormal process. The Turnbull & Wakeman (1991) paper matches the first two moments of the arithmetic average asset price to a lognormal distribution and an approximate Asian option price is obtained. The Brigo, Mercurio, Rapisarda & Scotti (2004) paper [10] uses a moment-matching approach to price basket-options. Stace (2007) [80] matches the first two moments of the VWAP to a lognormal distribution. Novikov et al. (2010) [62] matches the first two moments of the VWAP to a Generalised Inverse Gaussian distribution (GIG). In this thesis, we studied the methods in the last two papers. First, we approximate the distribution to the VWAP by a lognormal process $\tilde{S}^{LN}(t)$. Approximating to this distribution has the advantage of leading to an analytically tractable problem. Under the objective measure, the dynamics of the process $\tilde{S}^{LN}(t)$ is given by

$$d\tilde{S}(t) = \tilde{\mu}\tilde{S}^{LN}(t)dt + \tilde{\sigma}\tilde{S}^{LN}(t)dW(t)$$

with $\tilde{S}^{LN}(0) = S(0)$, $\tilde{\mu}$ the drift coefficient, $\tilde{\sigma}$ is the diffusion coefficient, and $W(t), 0 \leq t \leq T$, a Brownian motion on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, and $\tilde{\mathcal{F}}(t), 0 \leq t \leq T$, a filtration for $W(t)$ where $T > 0$. The parameters $\tilde{\mu}$ and $\tilde{\sigma}$ need to be found. The VWAP call option is a function of t and $\tilde{S}^{LN}(t)$, i.e. $C(t, \tilde{S}^{LN}(t))$. With taking the lognormal density as the state-price density function, we can obtain a closed-form analytic expression for the value of a VWAP option.

Second, we approximate the distribution to the VWAP by a GIG process $\tilde{S}^{GIG}(t)$. The GIG density is given by

$$p(x; a, b, p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} \exp\left\{-\frac{(ax + b/x)}{2}\right\}, \ x > 0$$

where $a > 0$, $b > 0$, p is a real number and K_p is a modified Bessel function of the second kind. The parameters a, b, p need to be found. Then similar to matching lognormal, we assume VWAP call option is a function of t and $\tilde{S}^{GIG}(t)$, i.e. $C(t, \tilde{S}^{GIG}(t))$. Then by using the GIG distribution as the state-price density function, we can obtain a closed-form analytic expression for the value of a VWAP option.

5.3 The moment-matching technique

The moment matching approach is a method whereby a number of moments of the process A_t at the time T are set equal to the corresponding moments of a candidate approximating process. The resulting set of equations then allows us to derive the parameters of the approximating process.
To approximate a non Gaussian distribution by a lognormal, we choose parameters $\tilde{\mu}, \tilde{\sigma}$ of the lognormal such that the lognormal moments match our VWAP moments. In other words, to match A_t to a lognormal process $\tilde{S}_t$ with drift $\tilde{\mu}$ and volatility $\tilde{\sigma}$ we require only the first two moments of A_T. We recall that the mean and the variance of $\tilde{S}_t$ are given by

$$\mathbb{E}\left(\tilde{S}_t\right) = \tilde{S}_0 e^{\tilde{\mu}t}, \ \mathrm{Var}\left(\tilde{S}_t\right) = \tilde{S}_0{}^2 e^{2\tilde{\mu}t}\left(e^{\tilde{\sigma}^2 t} - 1\right).$$

Making the substitutions $\mathbb{E}\left(\tilde{S_T}\right) = \mathbb{E}\left(A_T\right)$ and $\mathbb{E}\left(\tilde{S_T}^2\right) = \mathbb{E}\left(A_T^2\right)$ allows us to obtain the parameter values $\tilde{\mu}$ and $\tilde{\sigma}$ since

$$\tilde{\mu}(t) = \frac{1}{t}\log\frac{\mathbb{E}(\tilde{S}_t)}{S(0)} \tag{5.5}$$

$$\tilde{\sigma}(t) = \sqrt{\frac{1}{t}\log\frac{\mathrm{Var}\left(\tilde{S}_t\right) + \mathbb{E}(\tilde{S}_t)^2}{\mathbb{E}(\tilde{S}_t)}} \tag{5.6}$$

Now for any given time T one can find $\tilde{\mu}$ and $\tilde{\sigma}$ which matches the final distribution of the VWAP to a lognormal distribution, these are given by $\tilde{\mu}(T)$ and $\tilde{\sigma}(T)$. This implies that we now have the parameters $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ for the process $\tilde{S}_t$ at all times. Nevertheless, the approach in obtaining the VWAP moments will be described in the next section. Another candidate distribution could be the Generalised Inverse Gaussian distribution (GIG). The choice of approximating with the GIG is motivated by past research on the pricing of Asian options [17; 51]. It is known in the Mathematical Finance literature that the flexible semi-heavy tailed distribution is a natural choice for approximating Asian options on stocks with large volatilities.

The GIG distribution has the density function as in (2.5.4) Its i^{th} moment is given by

$$m_i = \left(\frac{b}{a}\right)^{i/2}\frac{K_{p+i}\left(\sqrt{ab}\right)}{K_p(\sqrt{ab})}. \tag{5.7}$$

To match A_t to a GIG process with three unknown parameters a, b and p. We require the first three VWAP moments $\mathbb{E}\left(A_T\right)$, $\mathbb{E}\left(A_T^2\right)$ and $\mathbb{E}\left(A_T^3\right)$, the matching of moments gives a system of three nonlinear equations

$$m_i = \mathbb{E}\left(A_T^i\right), \ i = 1, 2, 3$$

with three unknowns parameters a, b and p to be found.

5.4 Deriving Analytical Moments

In this section, analytical formulae for the first and second moments of the VWAP are derived via the calculation of the Laplace transform of the integral of the squared Ornstein-Uhblenbeck process. This methodology is chosen based on its convenience and simplicity. Calculations of this type (which are based only on using the Girsanov transformation and do not involve solving any PDEs and ODEs) have been done in the context of the calibration of an Ornstein-Uhlenbeck process [61].

5.4.1 First moment

The first moments is given by the following

Proposition 5.1.

$$\mathbb{E}(A_T) = -S_0 e^{\mu t} \int_0^T \left(\int_0^\infty \frac{\partial}{\partial z}\bigg|_{z=0} \phi(z,0,q)dq \right) dt \tag{5.8}$$

The mean of the VWAP process is computed in the following manner

$$\begin{aligned}
\mathbb{E}\left(A_T\right) &= \mathbb{E}\left(\frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt}\right) \\
&= \int_0^T \mathbb{E}\left(\frac{S_t U_t}{V_T}\right) dt \\
&= \int_0^T \mathbb{E}\left(\frac{S_0 e^{mt+\sigma_{BM}(\rho\widehat{W}_t+\sqrt{1-\rho^2}\widehat{W}_t)+X_t} U_t}{\int_0^T U_t dt}\right) dt \\
&= S_0 \int_0^T \mathbb{E} e^{X_t+mt+\sigma_{BM}\sqrt{1-\rho^2}\widehat{W}_t} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t} U_t}{V_T}\right) dt
\end{aligned}$$

The next step consists of elimination of the term $e^{\sigma_{BM}\rho\widehat{W}_t}$ using the change of measure. For convenience we define

$$\begin{aligned}
\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} &= \mathcal{E}_t\left(\int_0^{\cdot} \lambda dW_s\right), \ \mathbb{P}-\text{a.s.} \\
&= \exp\left\{\int_0^t \lambda dW_s - \frac{1}{2}\int_0^t |\lambda|^2 ds\right\}
\end{aligned}$$

where W_t is a $\mathbb{P}$ Brownian Motion and λ is constant satisfying

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T |\lambda|^2 dt\right)\right) < \infty.$$

Hence, we define the Radon Nikodym derivative as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = e^{\rho\sigma_{BM}\widehat{W}_T - \frac{\rho^2}{2}\sigma_{BM}^2 T}$$

is an exponential martingale and therefore by Girsanov theorem,

$$dW_t^{\mathbb{Q}} = dW_t + \rho\sigma_{\mathrm{BM}} dt$$

$$\mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\right) = \mathbb{E}^{\mathbb{Q}}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|\mathcal{F}_t\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}e^{-\rho\sigma_{BM}\widehat{W}_t+\frac{\rho^2}{2}\sigma^2_{BM}t}\right)$$

$$= e^{\frac{1}{2}\rho^2\sigma^2_{BM}t}\mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right) \tag{5.9}$$

Hence we have,

$$\mathbb{E}\left(A_T\right) = S_0\int_0^T \mathbb{E}e^{X_t+mt+\sigma_{BM}\sqrt{1-\rho^2}\widehat{W}_t}e^{\frac{1}{2}\rho^2\sigma^2_{BM}t}\mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right)dt$$

$$= S_0 e^{\log\left(\frac{1}{1-\theta\nu+\frac{\sigma^2}{2}\nu}\right)^{\frac{t}{\nu}}}e^{\frac{1}{2}\sigma^2_{BM}(1-\rho^2)t}e^{\mu t}e^{-\frac{1}{2}\sigma^2_{BM}t}e^{-\log\left(\frac{1}{1-\theta\nu+\frac{\sigma^2}{2}\nu}\right)^{\frac{t}{\nu}}}e^{\frac{1}{2}\rho^2\sigma^2_{BM}t}\int_0^T \mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right)dt$$

$$= S_0 e^{t(\mu+\frac{1}{2}\sigma^2_{BM}(1-\rho^2-1))}e^{\frac{1}{2}\rho^2\sigma^2_{BM}}\int_0^T \mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right)dt$$

$$= S_0 e^{\mu t}\int_0^T \mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right)dt \tag{5.10}$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation with respect to measure $\mathbb{Q}$. For convenience, write

$$V_T = \int_0^T U_t dt$$

We need to find $\mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right)$. In refer to paper [62], this expectation can be found by computing the Laplace transform of the integral of the squared Ornstein-Uhblenbeck process.
Defined the joint Laplace transform as

$$\phi(z,r,q) = \mathbb{E}^{\mathbb{Q}}(\exp\{-zU_t - rU_s - qV_T\}) \tag{5.11}$$

and assuming that

$$\mathbb{E}^{\mathbb{Q}}(U_t/V_T) < \infty$$

we can compute $\mathbb{E}(A_T)$ as follows. First we note that

$$\frac{\partial}{\partial z}\Phi(z,0,q)\bigg|_{z=0} = \mathbb{E}^{\mathbb{Q}}\left(\frac{\partial}{\partial z}e^{-zU_t-qV_T}\right)\bigg|_{z=0} = -\mathbb{E}^{\mathbb{Q}}\left(U_t e^{-qV_T}\right). \tag{5.12}$$

Integrating both sides of equation (5.12) we have

$$-\int_0^\infty \frac{\partial}{\partial z}\bigg|_{z=0}\phi(z,0,q)dq = -\frac{\partial}{\partial z}\bigg|_{z=0}\int_0^\infty \phi(z,0,q)dq = -\frac{\partial}{\partial z}\bigg|_{z=0}\int_0^\infty \mathbb{E}^{\mathbb{Q}}(e^{-zU_t-q\int_0^T U_t dt})dq$$

$$= \int_0^\infty \mathbb{E}^{\mathbb{Q}}\left(U_t e^{-qV_T}\right)dq = \mathbb{E}^{\mathbb{Q}}\left(\frac{U_t}{V_T}\right) \tag{5.13}$$

Hence the first moment can be represented now as the following

$$\mathbb{E}(A_T) = -S_0 e^{\mu t}\int_0^T\left(\int_0^\infty \frac{\partial}{\partial z}\bigg|_{z=0}\phi(z,0,q)dq\right)dt$$

Next we find $\phi(z,0,q)$

$$\begin{aligned}\phi(z,0,q) &= \mathbb{E}^{\mathbb{Q}}(\exp(-zU_t - q\int_0^T U_t dt)) \\ &= \exp\{-z\delta - q\delta T\}\,\Psi(z,q)\end{aligned}$$

where

$$\Psi(z,q) = \mathbb{E}^{\mathbb{Q}}(\exp\{-zX_t^2 - q\int_0^T X_t^2 dt\}).$$

The next step consists of elimination of the term $\int_0^T X_t^2 dt$ using a change of measure. For convenience we define the stochastic exponential

$$\eta_T(\lambda) = \exp\{-\lambda\int_0^T W_t dW_t - \lambda^2/2\int_0^T W_t^2 dt\}$$

where W_t is a standard Brownian motion. Using the Girsanov theorem (see details in [44]), we obtain

$$\begin{aligned}\Psi(z,q) &= \mathbb{E}^{\mathbb{Q}}(\eta_T(\lambda)\exp\left\{-z(a+vW_t)^2 - q\int_0^T (a+vW_t)^2 dt\right\}) \\ &= \mathbb{E}^{\mathbb{Q}}(\exp\left\{-z(a+vW_t)^2 - q\int_0^T (a^2+2vaW_t)dt - \lambda\int_0^T W_t dW_t - (\lambda^2/2+qv^2)\int_0^T W_t^2 dt\right\}).\end{aligned}$$

Set

$$\kappa = \sqrt{\lambda^2 + 2qv^2}.$$

Since $\int_0^T W_t dW_t = (W_T^2 - T)/2$ we have

$$\Psi(z,q) = \mathbb{E}^{\mathbb{Q}}(\eta_T(\kappa)\exp\left\{-z(a+vW_t)^2 - q\int_0^T (a^2+2vaW_t)dt - \frac{(\lambda-\kappa)(W_T^2-T)}{2}\right\})$$

and using the Girsanov theorem again,

$$= \mathbb{E}^{\mathbb{Q}}(\exp\left\{-z(a+vY_t)^2 - q\int_0^T (a^2+2vaY_t)dt - \frac{(\lambda-\kappa)(Y_T^2-T)}{2}\right\})$$

where Y_t is a standard Ornstein-Uhlenbeck process with parameter κ i.e. $Y_t = e^{-\kappa t}\int_0^t e^{\kappa s}dW_s$. After some simplifications we obtain

$$\Psi(z,q) = \exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\gamma(z,q) \tag{5.14}$$

where

$$\gamma(z,q) = \mathbb{E}^{\mathbb{Q}}(\exp\left\{-2zvaY_t - 2qva\int_0^T Y_s ds - zv^2Y_t^2 + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}).$$

To compute $\gamma(z,q)$, we condition over the filtration $\mathcal{F}_t$,

$$\begin{aligned}\gamma(z,q) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-2zvaY_t - 2qva\int_0^T Y_s ds - zv^2Y_t^2 + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\middle|\,\mathcal{F}_t\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{\xi}\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-2qva\int_t^T Y_s ds + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\middle|\,\mathcal{F}_t\right)\right]\end{aligned} \tag{5.15}$$

where $\xi = -2zvaY_t - 2qva\int_0^t Y_s ds - zv^2 Y_t^2$ and is $\mathcal{F}_t$-measurable. Using the fact that Y_t is a Markov process, the inner expectation of equation (5.15) can be expressed as

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-2qva\int_t^T Y_s ds + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\middle|\, Y_t\right). \tag{5.16}$$

Set $X_1 = a\int_t^T Y_s ds$, $X_2 = Y_T$, $X_3 = Y_t$ and $\sigma_{ij} = \text{Cov}(X_i, X_j)$ for $i,j \in \{1,2,3\}$ (see the appendix A.2 for the calculation of the covariances). Because Y_t is an Ornstein-Uhlenbeck process, X_1, X_2 and X_3 are Gaussian random variables and so together they form a multivariate normal distribution. Then the distribution of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ given $X_3 = z$ is a multivariate normal distribution with the mean vector and covariance matrix given by

$$\mu = \begin{bmatrix} \mu_1 + \dfrac{\sigma_{13}}{\sigma_{33}}(z-\mu_3) \\ \mu_2 + \dfrac{\sigma_{23}}{\sigma_{33}}(z-\mu_3) \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_{11} - \dfrac{\sigma_{13}^2}{\sigma_{33}} & \sigma_{12} - \dfrac{\sigma_{13}\sigma_{23}}{\sigma_{33}} \\ \sigma_{12} - \dfrac{\sigma_{13}\sigma_{23}}{\sigma_{33}} & \sigma_{22} - \dfrac{\sigma_{23}^2}{\sigma_{33}} \end{bmatrix}$$

respectively. So to compute the conditional expectation of equation (5.16), we can find

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-2qvX_1 + \frac{(\kappa-\lambda)X_2^2}{2}\right\}\,\middle|\, X_3 = z\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-2qvx + \frac{(\kappa-\lambda)y^2}{2}\right\} f_{X_1,X_2|X_3}(x,y\mid z)\, dxdy \end{aligned} \tag{5.17}$$

where $f_{X_1,X_2|X_3}(x,y\mid z)$ is the density function of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ given $X_3 = z$.

Computation of the double integral of equation (5.17) essentially requires us to solve

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-Ax^2 - By^2 + Cx + Dy + Fxy + G\right\} dxdy \tag{5.18}$$

where A, B, C, D, F and G are constants. Under the condition $F^2 < 4AB$ the solution to equation (5.18) is

$$2\pi \exp\left\{\frac{BC^2 + D(AD + CF)}{4AB - F^2} + G\right\}(4AB - F^2)^{-1/2}.$$

Using this result, in addition to performing a number of symbol manipulations in Mathematica, we can rewrite equation (5.1) as

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-2qva\int_t^T Y_s ds + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\middle|\, \mathcal{F}_t\right) = \exp\left\{HY_t^2 + JY_t + L\right\}$$

where the constants H, J and L are known. Mathematica expressions for these constants are too long to reproduce here; See Appendix for Mathematica codes. This in turn allows us to express $\gamma(z,q)$ of equation (5.15) as another double integral of the same form as equation (5.18). This leads to a closed-form expression for the joint Laplace transform $\Phi(z,0,q)$ where its partial derivative with respect to z may be computed analytically.

Remark. The Laplace transform of V_T given by $\Phi(0,0,q)$ was originally derived in [61]. In particular, the following expression was obtained in [61] (see also Section 17.3 in [61]) for the case $a = 0$

and $q \geq 0$:

$$g(q) = \mathbb{E}^{\mathbb{Q}} \exp\left(-qv^2 \int_0^T \xi_s^2 ds\right) = [\frac{2\kappa e^{\lambda T}}{(\kappa - \lambda)e^{-\kappa T} + (\kappa + \lambda)e^{\kappa T}}]^{1/2},$$

where the process ξ_s is defined in equation (5.1) and $\kappa = \sqrt{\lambda^2 + 2qv^2}$. In view of Andersen's Lemma ([44], see also Section 2.10 in [36]) and taking into account equation (5.1) we have for any $X_0 = a$ and $x > 0$

$$\mathbb{P}\{\int_0^T X_s^2 ds < x\} \leq \mathbb{P}\{v^2 \int_0^T \xi_s^2 ds < x\}.$$

This implies the following estimate for any $p > 0$

$$\mathbb{E}^{\mathbb{Q}}(V_T^{-p}) = \frac{1}{\Gamma(p)} \int_0^\infty q^{p-1} \Phi(0,0,q) dq \leq \frac{1}{\Gamma(p)} \int_0^\infty q^{p-1} g(q) dq.$$

Since $g(q) = O(e^{-\kappa T})$ as $q \to \infty$ this estimate implies

$$\mathbb{E}^{\mathbb{Q}}(V_T^{-p}) < \infty.$$

When $\delta > 0$ this result is, of course, trivial. Since U_t is a shifted squared Gaussian process we have also $\mathbb{E}^{\mathbb{Q}}(U_t^p) < \infty$ for any $p > 0$. Using the Hölder inequality we obtain that for any $p > 0$

$$\mathbb{E}^{\mathbb{Q}}(U_t^p / V_T^p) < \infty$$

and so condition $\mathbb{E}^{\mathbb{Q}}(U_t / V_T) < \infty$ holds.

Since σ_{BM} does not enter into the computation of $\mathbb{E}(A_T)$, we have the following remark.

Remark 5.1.

$$\mathbb{E}(A_T^{\text{VG}}) = \mathbb{E}(A_T^{\text{VG}}) = S_0 e^{\mu t}$$

5.4.2 Second moment

Using the same technique as the derivation of the first VWAP moment, the VWAP second moments is given by

Proposition 5.2.

$$\mathbb{E}\left(A_T^2\right) = \int_0^T \int_0^T \int_0^\infty q S_0^2 e^{(t \wedge s)\tilde{\psi}(2)} e^{|t-s|\tilde{\psi}(1)} e^{\frac{1}{2}\sigma_{BM}^2 (s+t)} \left(\frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\bigg|_{z=r=0}\right) dq dt ds. \tag{5.19}$$

Given the Laplace Transform $\phi(z,r,q)$ in equation (5.11) where

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{U_t U_s}{V_T^2}\right) = \int_0^\infty q \frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\bigg|_{z=r=0} dq \tag{5.20}$$

$$\tilde{\psi}(n) = nm + \frac{1}{2}n^2 \tilde{\sigma}_{BM}^2 + \log\left(\frac{1}{1 - n\theta\nu + \frac{(n\sigma_{VG})^2}{2}\nu}\right) \bigg/ \nu$$

$$\tilde{\sigma}_{BM} = \sigma_{BM}(1 - \rho^2)$$

To prove this result. First we need the following lemma 5.3 and proposition 5.4.

Lemma 5.3.

$$\mathbb{E}(S_t^n) = S_0^n e^{t\psi(n)} \tag{5.21}$$

where

$$\psi(n) = nm + \frac{1}{2}n^2\sigma_{BM}^2 + \log\left(\frac{1}{1 - n\theta\nu + \frac{(n\sigma_{VG})^2}{2}\nu}\right)\Big/\nu$$

Proof. For any $n > 0$,

$$\mathbb{E}(S_t^n) = \mathbb{E}((S_0 e^{L_t})^n) = S_0^n \mathbb{E}e^{nL(t)} = S_0^n e^{t\psi(n)}$$

Now $L_t = mt + \sigma_{BM}\bar{W}_t^{(2)} + X_t^{VG}$

$$\begin{aligned}
\mathbb{E}(S_t^n) &= \mathbb{E}((S_0 e^{mt + \sigma_{BM} W_t + X_t})^n) \\
&= S_0^n e^{nmt} e^{\frac{1}{2}n^2\sigma_{BM}^2 t}\mathbb{E}e^{nX_t} \\
&= S_0^n e^{nmt} e^{\frac{1}{2}n^2\sigma_{BM}^2 t}\left(\frac{1}{1 - n\theta\nu + \frac{(n\sigma_{VG})^2}{2}\nu}\right)^{\frac{t}{\nu}} \\
&= S_0^n e^{nmt + \frac{1}{2}n^2\sigma_{BM}^2 t + \log\left(\left(\frac{1}{1 - n\theta\nu + \frac{(n\sigma_{VG})^2}{2}\nu}\right)^{\frac{t}{\nu}}\right)} \\
&= S_0^n e^{t\left(nm + \frac{1}{2}n^2\sigma_{BM}^2 + \log\left(\frac{1}{1 - n\theta\nu + \frac{(n\sigma_{VG})^2}{2}\nu}\right)\Big/\nu\right)} \\
&= S_0^n e^{t\psi(n)}
\end{aligned}$$

where

$$\psi(n) = nm + \frac{1}{2}n^2\sigma_{BM}^2 + \log\left(\frac{1}{1 - n\theta\nu + \frac{(n\sigma_{VG})^2}{2}\nu}\right)\Big/\nu \tag{5.22}$$

□

Set $n = 1$ in equation (5.22)

$$\Longrightarrow \mathbb{E}S_t = S_0 e^{t\psi(1)} = S_0 e^{t\left(m + \frac{\sigma_{BM}^2}{2} + \log\left(\frac{1}{1 - \theta\nu + \frac{(\sigma_{VG})^2}{2}\nu}\right)\Big/\nu\right)} = S_0$$

iff

$$m = -\frac{1}{2}\sigma_{BM}^2 - \log\left(\frac{1}{1 - \theta\nu + \frac{(\sigma_{VG})^2}{2}\nu}\right)\Big/\nu \quad \text{Assume no growth in stock price}$$

In general, we add the expected return μ of S_t to m, hence

$$m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log\left(\frac{1}{1 - \theta\nu + \frac{(\sigma_{VG})^2}{2}\nu}\right)\Big/\nu$$

Proposition 5.4. *Let S_t and S_s be any two Geometric Lévy process of the form $S_t = S_0 e^{L_t}$, where*

$L_t = mt + \sigma_{BM} W_t + X_t$ *then*

$$\mathbb{E}(S_t S_s) = S_0^2 \mathbb{E} e^{2L_s} \mathbb{E} e^{L_t - L_s} = S_0^2 e^{(t\wedge s)\psi(2)} e^{|t-s|\psi(1)} \tag{5.23}$$

Now we are in a position to prove proposition 5.2, the idea is the following.

Proof. Recall in the model setup, $\widetilde{W}_t \perp \widehat{W}_t$, The VWAP second moment is given by

$$\begin{aligned}
\mathbb{E}\left(A_T^2\right) &= \mathbb{E}\frac{(\int_0^T S_t U_t dt)^2}{(\int_0^T U_t dt)^2} \\
&= \int_0^T \int_0^T \mathbb{E}\left(\frac{S_0 e^{mt+\sigma_{BM}(\rho\widehat{W}_t+\sqrt{1-\rho^2}\widetilde{W}_t)+X_t} U_t S_0 e^{ms+\sigma_{BM}(\rho\widehat{W}_s+\sqrt{1-\rho^2}\widetilde{W}_s)+X_s} U_s}{V_T^2}\right) dtds \\
&= \int_0^T \int_0^T \mathbb{E} S_0^2 e^{mt+X_t+\sigma_{BM}\sqrt{1-\rho^2}\widetilde{W}_t+ms+X_s+\sigma_{BM}\sqrt{1-\rho^2}\widetilde{W}_s} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\right) dtds \\
&= \int_0^T \int_0^T \mathbb{E} S_0^2 e^{\tilde{L}_t+\tilde{L}_s} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\right) dtds \\
&= \int_0^T \int_0^T S(0) \mathbb{E} e^{2\tilde{L}_s} \mathbb{E} e^{\tilde{L}_t-\tilde{L}_s} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\right) dtds \\
&= \int_0^T \int_0^T S(0) \mathbb{E} e^{2\tilde{L}_s} \mathbb{E} e^{\tilde{L}_t-\tilde{L}_s} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\right) dtds
\end{aligned}$$

where $\tilde{L}_t = mt + \tilde{\sigma}_{BM} \widetilde{W}_t + X_t^{VG}$, $\tilde{\sigma}_{BM} = \sigma_{BM}\sqrt{1-\rho^2}$. Now apply lemma 5.3 and proposition 5.4 with taking $\tilde{\sigma}^2 = \sigma_{BM}^2(1-\rho^2)$

$$\mathbb{E}\left(A_T^2\right) = \int_0^T \int_0^T S_0^2 e^{(t\wedge s)\tilde{\psi}(2)} e^{|t-s|\tilde{\psi}(1)} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\right) dtds \tag{5.24}$$

where $\tilde{\psi}(n) = nm + \frac{1}{2}n^2\tilde{\sigma}_{BM}^2 + \ln\left(\frac{1}{1-n\theta\nu+\frac{n^2\sigma_{VG}^2}{2}}\right)\Big/\nu$ The next step consists of eliminating of the term $e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)}$ using the change of measure. Define the Radon Nikodym derivative as

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = e^{\rho\sigma_{BM}(\widehat{W}_T+\widehat{W}_S)-\frac{\rho^2}{2}\sigma_{BM}^2(S+T)}, \ \mathbb{P}-\text{a.s.}$$

$$\begin{aligned}
\mathbb{E}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\right) &= \mathbb{E}^{\mathbb{Q}}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2}\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}(t)}\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\frac{e^{\sigma_{BM}\rho(\widehat{W}_t+\widehat{W}_s)} U_t U_s}{V_T^2} e^{-\rho\sigma_{BM}(\widehat{W}_t+\widehat{W}_s)+\frac{\rho^2}{2}\sigma_{BM}^2(s+t)}\right) \\
&= e^{\frac{1}{2}\rho^2\sigma_{BM}^2(s+t)} \mathbb{E}^{\mathbb{Q}}\left(\frac{U_t U_s}{V_T^2}\right)
\end{aligned} \tag{5.25}$$

Given the Laplace transform $\Phi(z,r,q)$ in equation (5.11) we can compute $\mathbb{E}\left(A_T^2\right)$ as follows:

$$\begin{aligned} \frac{\partial}{\partial z}\Phi(z,r,q)\bigg|_{z=0} &= -\mathbb{E}^{\mathbb{Q}}\left(U_t e^{-zU_t - rU_s - qV_T}\right)\bigg|_{z=0} \\ \Longrightarrow \frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\bigg|_{z=r=0} &= \mathbb{E}^{\mathbb{Q}}\left(U_t U_s e^{-zU_t - rU_s - qV_T}\right)\bigg|_{z=r=0}. \end{aligned} \tag{5.26}$$

Now multiply both sides of equation (5.26) by q and integrating with respect to q over $[0,\infty)$:

$$\begin{aligned} q\int_0^\infty \left(\frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\bigg|_{z=r=0}\right)dq &= q\int_0^\infty \mathbb{E}^{\mathbb{Q}}\left(U_t U_s e^{-qV_T}\right)dq \\ &= \mathbb{E}^{\mathbb{Q}}\left(U_t U_s \int_0^\infty q e^{-V_T q} dq\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left(\frac{U_t U_s}{V_T^2}\right). \end{aligned}$$

So we have

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{U_t U_s}{V_T^2}\right) = \int_0^\infty q \frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\bigg|_{z=r=0} dq \tag{5.27}$$

and so the second moment is given by

$$\mathbb{E}\left(A_T^2\right) = \int_0^T \int_0^T \int_0^\infty q S_0^2 e^{(t\wedge s)\tilde{\psi}(2)} e^{|t-s|\tilde{\psi}(1)} e^{\frac{1}{2}(\rho\sigma_{BM})^2(s+t)} \left(\frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\bigg|_{z=r=0}\right) dq dt ds. \tag{5.28}$$

Further calculations of $\Phi(z,r,q)$ are similar to the case $\Phi(z,0,q)$ and thus are omitted here. We must note that all our analytical results have been implemented in the Mathematica software package and fully verified using Monte Carlo simulations (see Chapter 7). □

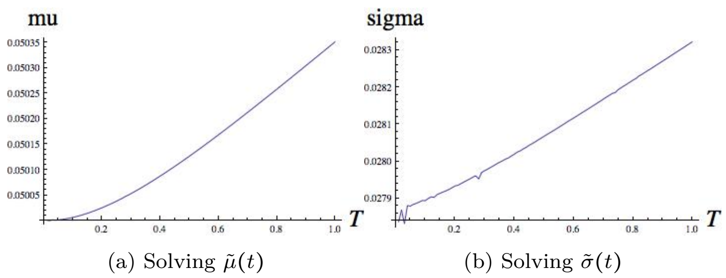
5.4.3 Pricing

To find the price for the VWAP call option call option maturing at T, with strike at K, with the terminal payoff function

$$C_T = (A_T - K)^+$$

using the moment matching technique, we need only the values of lognormal parameters $\tilde{\mu}$ and $\tilde{\sigma}$ at the terminal time, i.e. $\tilde{\mu}(T)$ and $\tilde{\sigma}(T)$. To obtain these two parameters, first, the analytical moments of VWAP derived in (5.4.2) and (5.4.1) are solved for each time over the interval $[0,1]$. Then based on the postulation that the moments of VWAP represent moments of the matching distribution (lognormal in this case), $\tilde{\mu}$ and $\tilde{\sigma}$ are inverted from the analytical moments.

Fig.5.1 illustrates how $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ evolve over time (See Appendix B for Mathematica codes).



(a) Solving $\tilde{\mu}(t)$ (b) Solving $\tilde{\sigma}(t)$

Figure 5.1: Solving $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ by computing VWAP moments.

We integrate against the state price density to obtain the option price. We consider two different processes to match. The first process is the well known lognormal process with two parameters, i.e.

$$\widetilde{A}_T \sim \text{lognormal}(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2, \tilde{\sigma}^2) \Longleftrightarrow \widetilde{A}_1 = e^{\tilde{\sigma} W_1 + \tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2}, \quad W_1 \sim N(0,1).$$

Following papers [79; 80; 62], we start by postulating that the process of VWAP represents an effective lognormal process and then match the first two moments:

$$\mathbb{E}A_T = \mathbb{E}(\widetilde{A}_T) = e^{\frac{1}{2}\tilde{\sigma}^2 T + (\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)} T = e^{\tilde{\mu}T}$$
$$\mathbb{E}A_T^2 = \mathbb{E}(\widetilde{A}_T^2) = e^{2\tilde{\sigma}^2 + 2(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)} = e^{2\tilde{\mu}T + \tilde{\sigma}^2 T}$$

In the absence of arbitrage, the risk-neutral price of a VWAP call option is the expected discounted payoff, which is simply a regular Riemann integral, i.e.

$$\ln \widetilde{A}(T) \sim N(\ln \widetilde{A}(0) + (\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)T, \tilde{\sigma}^2 T)$$

$$C_0 = \mathbb{E}(e^{-rT}(\widetilde{A}_T - K)^+) = \int_{\ln K}^{\infty} e^{-rT}(e^z - K)\mathbb{I}_{\{e^z > K\}} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(z - \ln S(0) - (r - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}} dz$$

The second process we consider is the so-called generalised inverse gaussian (GIG) process (which is a popular model in modern actuarial risk theory, see [58]). In this case we first postulate that the process of VWAP represents an GIG process,

$$\widetilde{A}_T \sim \text{GIG}\left(\frac{\sqrt{a}K_{p+1}(\sqrt{ab})}{\sqrt{a}K_p(\sqrt{ab})}, (\frac{b}{a})\left(\frac{K_{p+2}(\sqrt{ab})}{K_p(\sqrt{ab})} - \left(\frac{K_{p+1}(\sqrt{ab})}{K_p(\sqrt{ab})}\right)^2\right)\right)$$

Since the GIG process is characterized by three parameters, we match the first three moments, i.e.

$$\mathbb{E}A_T = \mathbb{E}(\widetilde{A}_T) = m_1^{(GIG)}$$
$$\mathbb{E}A_T^2 = \mathbb{E}(\widetilde{A}_T^2) = m_2^{(GIG)}$$
$$\mathbb{E}A_T^3 = \mathbb{E}(\widetilde{A}_T^3) = m_3^{(GIG)}$$

where $m_i^{(GIG)}$ are as in equation (5.7). Then similarly, the VWAP option price is obtained by computing the following regular Riemann integral (which is the discounted expected payoff), i.e.

$$C_0 = \mathbb{E}(e^{-rT}(\widetilde{A}_T - K)^+) = \int_{\ln K}^{\infty} e^{-rT}(A_T - K)\mathbb{I}_{\{A_T > K\}} \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-(ax + b/x)/2} dx$$

For illustration purposes, we consider the following parameter values for both lognormal and GIG approximations:

$$r = 0, \quad S_0 = 110, \quad K = 100, \quad r = 0, \quad T = 1, \quad \rho = 0$$

$$\mu = 0.1, \quad \sigma_{\text{BM}} = 0.1, \quad \theta = 0.14, \quad \nu = 0.1 \quad \sigma_{VG} = 0.1,$$

$$m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log\left(\frac{1}{1 - \theta\nu + \frac{(\sigma_{VG})^2}{2}\nu}\right) \Big/ \nu$$

$$\Delta\gamma^- \sim \Gamma(\Delta t \frac{1}{\nu}, \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma_{\text{VG}}^2 \nu}{2}} - \frac{\theta\nu}{2})$$

$$\Delta\gamma^+ \sim \Gamma(\Delta t\frac{1}{\nu}, \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma_{\text{VG}}^2\nu}{2}} + \frac{\theta\nu}{2})$$

and our parameters choice yields results as shown in Table 5.1

Table 5.1: Numerical values of call price and Monte Carlo simulation of call price for varying stock price volatility value σ_{BM}.

σ_{BM}	Monte Carlo	lognormal approximation	GIG approximation
0.1	15.89	15.68	15.83
0.2	16.64	16.11	16.59
0.3	18.07	17.61	18.00
0.4	19.89	19.61	19.79
0.5	21.92	21.82	21.76

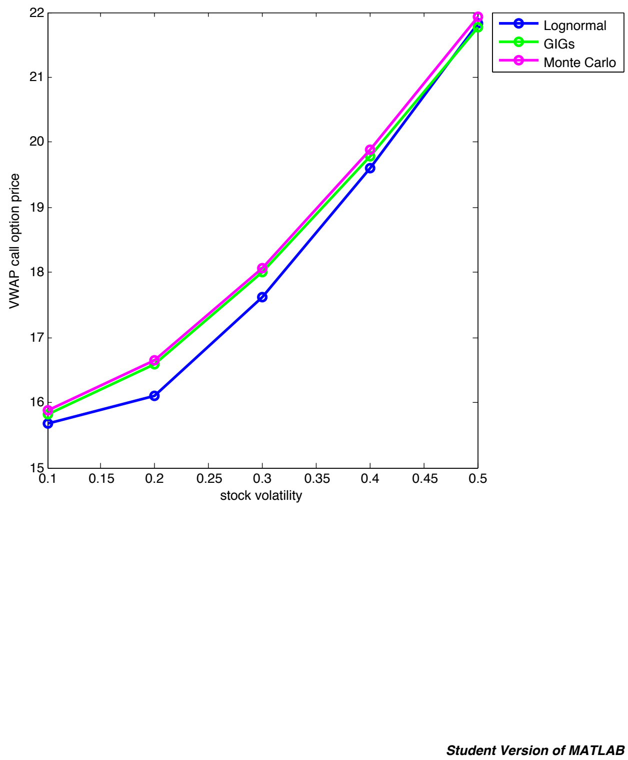


Figure 5.2: Call option prices for $K = 100$ and different stock price volatility σ.

Chapter 6

Monte Carlo Simulation

This chapter develops Monte Carlo methods to price the VWAP options. Methods developed in previous chapters are benchmarked and compared to the results of this chapter. The simulations are carried out in MATLAB, which provides us a rich set of tools for carrying out this task.
In this chapter, we first review some basic mathematics facts in regard to MC and gain a theoretical understanding of the method. Formulation of the problem and the discretization methods used to simulate underlying processes of interest follow. Then, the VWAP options are priced and results can be found in table 5.1 in the last chapter.

Monte Carlo (MC) is a simulation method. It was initially applied to option pricing by Boyle in 1977. Nowadays, it has been more and more widely applied to price options with complicated structures. MC has been widely used in financial engineering when an analytic solution for one problem is not available. Its applications range from pricing, hedging, risk management, etc. Compared with other methodologies in option pricing, Monte Carlo simulation is straight forward and easy to implement. Though Monte Carlo simulation is often considered as the last resort[1] in pricing derivatives, for derivatives with very complicated payoff structures, it is often the only feasible approach for pricing purpose.

6.1 The Mathematics behind MC

Suppose we want to estimate some J, and we have

$$J = \mathbb{E}(g(\mathbf{X})) \tag{6.1}$$

where $g(\mathbf{X})$ is an arbitrary function such that $\mathbb{E}(|g(\mathbf{X})|) < \infty$, then we could generate n independent random variate $X_1, X_2, ..., X_n$ such that all X_i have the same distribution as X then, according to the Weak Law of Large Number (WLLN) The estimator of J is given by

$$J_n = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$$

In particular, if $\mathbf{X}$ has the pdf $q(x_1, ..., x_m)$ then

$$J = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} g(x_1, ..., x_m) q(x_1, ..., x_m) dx_1 ... dx_m$$

and so we have an algorithm for approximating multidimensional integrals.

[1] When no close-form solution is available.

Since $\mathbb{E}(|g(\mathbf{X})|) < \infty$, by the Kolmogorov's Strong Law of Large Numbers, a stronger form of convergence holds:

$$J_n := \frac{1}{n}\sum_{i=1}^{n} g(X_i) \underset{n\to\infty}{\overset{a.s.}{\to}} \mathbb{E}g(\mathbf{X})$$

$$\text{i.e.} \quad J_n \underset{n\to\infty}{\overset{a.s.}{\to}} J$$

To estimate accuracy of the approximation, we assume

$$Var(g(\mathbf{X})) := \sigma^2(g) < \infty$$

and note

$$Var(J_n) = \frac{Var(g(\mathbf{X}))}{n} = \frac{\sigma^2(g)}{n}$$

Since the Monte Carlo simulation is never exact, and one always has to take the standard error into account. It can be expressed as

$$s(\mathbf{X}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(g(X_i) - J_n)^2} \quad \text{or} \quad s(\mathbf{X}) = \sqrt{\frac{\sum_{i=1}^{n} g(X_i)^2}{n} - (J_n)^2}$$

And applying CLT we have the convergence $(J_n - J)\sqrt{n} \overset{d}{\to} N(0, \sigma^2(g))$. In particular, it implies that

$$|J_n - J| \le 3\frac{\sigma(g)}{\sqrt{n}} \quad \text{with} \quad \Pr\left\{|J_n - J| \le \frac{3\sigma(g)}{\sqrt{n}}\right\} \approx 0.997 \quad \text{assuming large } n \tag{6.2}$$

The constant $\sigma^2(g)$ is usually unknown but it can be estimated using Weak Law of Large Number (WLLN) by

$$\hat{\sigma}_n^2(g) := \frac{1}{n}\sum_{i=1}^{n} g^2(X^{(i)}) - (J_n)^2 \overset{\mathbb{P}}{\to} \mathbb{E}(g^2(\mathbf{X})) - J^2 = \sigma^2(g)$$

CLT also tells us that

$$\sqrt{n}\frac{(J_n - J)}{\sigma} \underset{n\to\infty}{\overset{a.s.}{\to}} N(0,1)$$

In other words, for large n we have

$$\frac{\sum_{i=1}^{n}(X_i - \mathbb{E}(\mathbf{X}))}{\sqrt{n}s(\mathbf{X})} \sim N(0,1) \quad \text{or} \quad \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mathbb{E}(\mathbf{X}) \sim N(0, \frac{\text{Var}(\mathbf{X})}{n})$$

One could say that $J_n - J$ is approximately a standard normal variable scaled by $\frac{s(\mathbf{X})}{\sqrt{n}}$, i.e. for large n we have

$$P\left\{J_n - z_{\frac{\alpha}{2}}\frac{s(\mathbf{X})}{\sqrt{n}} < J < J_n + z_{\frac{\alpha}{2}}\frac{s(\mathbf{X})}{\sqrt{n}}\right\} \approx 1 - \alpha$$

For example, an estimate of the 95%-confidance interval for J is given by

$$\left(J_n - 1.96\frac{s(\mathbf{X})}{\sqrt{n}}, J_n + 1.96\frac{s(\mathbf{X})}{\sqrt{n}}\right).$$

The estimation of J by J_n is referred as the crude Monte Carlo method. The disadvantage of the crude Monte Carlo method is its slow rate of convergence. Since for large n we have $s \approx \sigma$, we have to enlarge our n by a factor of 100 to achieve a reduction of the standard error $\frac{s(\mathbf{X})}{\sqrt{n}}$ of a factor 0.1. Thus more accuracy is ensured by more iterations.

6.2 The Monte Carlo Techniques Applied to Option Pricing

The value of a call option at time t is the discounted expected terminal payoff, i.e.

$$C(t, S(t)) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}((S(T) - K))^{+}$$

where the expectation is taken under the risk neutral measure. The approximation of the option price relies upon the Strong Law of Large Number (SLLN). To utilise $SLLN$, the underlying process, i.e. $S(t)$, is simulated from time 0 until time T for N times. For each of these trials, the payoff is computed and discounted by $e^{-r(T-t)}$ to form C_i, C is then approximated by $\hat{C}_N$, where

$$\hat{C}_N = \frac{1}{N}\sum_{i=1}^{N} C_i$$

Then by the SLLN, $\hat{C}_N$ converges to the true expectation as $N \to \infty$. Another issue that deserves some special attention is the sources of error that could occur. The first type of error arises from the *randomness of MC*. The second type of error arises from the *discretization of the price and volume processes*. Suppose we have a close form solution, we wish to identify discretisation bias versus random error, To find out how much of the discretisation error, one way is to keep running the Monte Carlo simulations until we see the evidence of the bias. But how do we identify such evidence of bias? It can be shown by the standard error, i.e. $\frac{\sigma}{\sqrt{N}}$, if it exceeds 2 , i.e. statistically significant, it could worth more effort to increase the number of time steps rather than increasing the number of simulations until standard error is brought down. Once the standard error is brought down, one can start increasing the number of MC runs. Some typical examples are the simulation of a Lognormal LIBOR model, which is outside the scope of this thesis. (We refer readers to Brigo & Mercurio (2006) [9])
In the case of VWAP, the explicit solutions are known for $S(t)$ and $U(t)$, we use the crude MC by discretizing the **solution** of the Stochastic Differential Equations with respect to the two processes. This algorithm to price European option via MC is summarised as follows.

- Simulate the underlying processes under the risk neutral measure N times,
- Calculate the discounted payoff for each of the simulations,
- Average the discounted payoffs, and
- Calculate the standard deviation of the solution.

To illustrate the Monte Carlo method, we first consider to price a standard European call that is determined by the terminal stock price $S(T)$. The price of the option in this case is independent to the evolution of $S(t)$ between time 0 and time T. Let $S(T)$ evolves as a GBM, i.e.

$$S(T) = S(0)\exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right), \quad \text{where} \quad Z \sim N(0,1)$$

Then a simple Monte Carlo algorithm to estimate $\mathbb{E}(e^{-rt}(S(T)-K)^{+})$ is the following.

for $i = 1, ..., N$ **do**
generate Z_i
set $S(T) = S(0)e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z_i}$
set $C_i = e^{-rT}(S(T) - K)^{+}$
end for
set $\hat{C}_N = (C_1 + ... + C_N)/N$

However, in valuing more complicated options using more complicated models of the dynamics of the underlying assets. It is often necessary to simulate a path over multiple intermediate date and not just at the initial and terminal date [34]. For instance, in the case of simulating Asian options and VWAP options, the price of the option depends on intermediate value of the terminal date. In the latter, the VWAP is obtained by weighted against the stock prices by their trade volume. Hence, the option price does depend on the intermediate value of both underlying stock price and volume. The following algorithm is an illustration of simulating M paths with N number of discretisation points.

for $i = 1, ..., M$ **do**
  for $i = 1, ..., N$ **do**
    generate Z_{ij}
    set $S_i(t_j) = S_i(t_{j-1})\exp\left((r - \frac{\sigma^2}{2})(t_j - t_{j-1}) + \sigma\sqrt{(t_j - t_{j-1})}Z_{ij}\right)$
  end for
  $\bar{S} = (S_i(t_1) + ... + S_i(t_N))/N$
  $C_i = e^{-rT}(\bar{S} - K)^+$
end for
set $\hat{C}_M = (C_1 + ... + C_M)/M$

6.3 Problem Formulation

We consider to price the VWAP call option

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(\frac{\int_0^T S(t)U(t)dt}{\int_0^T U(t)dt} - K\right)^+$$

The risk neutral dynamics of the state variables are described as the same way as in 5.1. Nevertheless, we restate as following

$$S(t) = S(0)e^{rt+mt+X^{VG}(t)+\sigma_{BM}\bar{W}^{(2)}(t)} \tag{6.3}$$

$$dX(t) = \lambda(a - X(t))dt + \sigma d\bar{W}^{(1)}(t) \tag{6.4}$$

$$U(t) = X(t)^2 + \delta \tag{6.5}$$

where $\bar{W}^{(2)}(t)$ is a Brownian motion correlated with the Brownian motion $\bar{W}^{(1)}(t)$ for the volume process U_t, X_t^{VG} is a variance gamma process independent of $\bar{W}^{(2)}(t)$, and $m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log(1 - \theta v - \frac{1}{2}\sigma_{VG}^2 v)/v$ is a drift to ensure $e^{-rt}S_t$ is a martingale. To price option via MC, we often need to simulate the SDE. Three situation we may have:

- Case 1: Explicit solution to the SDE is known.
- Case 2: Explicit solution to the SDE is unknown, parameters are constant. Example: The CIR model with constant parameters.
- Case 3: Explicit solution to the SDE is unknown, parameters are time varying. Example: The CIR model with time varying parameters.

For case 1, the simulation would be exact in the sense that the underlying dynamics can be exactly simulated at any given constant of time. For case 2, though no explicit solution to the SDE, we can still approximate the solution via a numerical scheme, i.e. an Euler or a Milstein scheme, however, both methods induce more discretisation errors. For case 3, a direct MC simulation would not work in general. But if the underlying dynamics are known, one can still simulate the SDE with a very slow convergence. In this thesis, case 1 applies as we already have explicit solutions for both $S(t)$ and $X(t)$. Hence, instead of simulate the SDE itself, we can just simulate the solution of the SDE, and the algorithm is an exact simulations. To summarise, the simulation of the system of equations is accomplished by

- Using exact solutions for $S(t)$ and $X(t)$ given by (equation (6.3) and equation (6.8)),
- Discretize the exact solutions of $S(t)$ and $X(t)$.

Different correlation levels between $\bar{W}^{(1)}(t)$ and $\bar{W}^{(2)}(t)$ are varied in the repetition of the numerical experiments to observe the effect of the correlation on the price of the option.

6.4 Simulation of Squared Ornstein-Uhblenbeck

We assume the stock trading volume follows an squared Ornstein-Uhblenbeck process

$$U(t) = X^2(t) + \delta \tag{6.6}$$

whereas $X(t)$ is defined as the unique strong solution of the following SDE:

$$dX(t) = \lambda(a - X(t))dt + \sigma_{\text{OU}} dW(t) \tag{6.7}$$

Here, W is a Brownian motion and λ, a are positive constants, and $\sigma_{\text{OU}} \in \mathbb{R}$. Notice that the drift in equation (6.7) is positive if $X(t) < a$ and negative if $X(t) > a$; thus, $X(t)$ is pulled toward level a, a property generally referred to as mean reversion.
Equation (6.7) is also called a Vasicek model and its unique strong solution is given by

$$\begin{aligned} X(t) &= e^{-\lambda t} X(0) + \lambda \int_0^t e^{-\lambda(t-s)} a ds + \sigma_{\text{OU}} \int_0^t e^{-\lambda(t-s)} dW(s) \\ &= a - e^{-\lambda t}(a - X(0)) + \sigma_{\text{OU}} \int_0^t e^{-\lambda(t-s)} dW(s) \\ &= e^{-\lambda t} X(0) + a(1 - e^{-\lambda t}) + \sigma_{\text{OU}} \int_0^t e^{-\lambda(t-s)} dW(s) \end{aligned} \tag{6.8}$$

Likewise, $\forall\ 0 < u < t$,

$$\begin{aligned} X(t) &= e^{-\lambda(t-u)} X(u) + \lambda \int_u^t e^{-\lambda(t-s)} a ds + \sigma_{\text{OU}} \int_u^t e^{-\lambda(t-s)} dW(s) \\ &= a - e^{-\lambda(t-u)}(a - X(u)) + \sigma_{\text{OU}} \int_u^t e^{-\lambda(t-s)} dW(s) \\ &= e^{-\lambda(t-u)} X(u) + a(1 - e^{-\lambda(t-u)}) + \sigma_{\text{OU}} \int_u^t e^{-\lambda(t-s)} dW(s) \end{aligned} \tag{6.9}$$

$X(t)$ defines an Ornstein-Uhblenbeck. Note that equations (6.8) and (6.9) is used for simulation in this thesis. From this it follows that, given $X(0)$ and given constant t, the value $X(t)$ is normally distributed with mean

$$\mathbb{E}(X(t)) = e^{-\lambda(t-u)} X(u) + \mu(s,t); \quad \mu(s,t) = a(1 - e^{t-u}) \tag{6.10}$$

and covariance function

$$Cov(X_u, X_t) = \frac{\sigma^2}{2\lambda} e^{-\lambda(u+t)} \left(e^{2\lambda(u \wedge t)} - 1\right) \tag{6.11}$$

When $u = t$ in equation (6.11), the variance is obtained as

$$Var(X_t) = \frac{\sigma_{\mathrm{OU}}^2}{2\lambda}(1 - e^{-2\lambda t}), \quad t \geq 0 \tag{6.12}$$

To simulate X at time $0 = t_0 < t_1 < ... < t_n$, one can set

$$\begin{aligned} X(t_i) &= e^{-\lambda(t_i - t_{i-1})} X(t_{i-1}) + \mu(t_{i-1}, t_i) + \sigma_{\mathrm{OU}}(t_{i-1}, t_i) Z_i \\ &= e^{-\lambda(t_i - t_{i-1})} X(t_{i-1}) + a(1 - e^{-\lambda(t_i - t_{i-1})}) + \sqrt{\frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda(t_i - t_{i-1})})} Z(t_i) \end{aligned} \tag{6.13}$$

with $Z_1, ..., Z_n$ are drawn standard normal random variates.

Our Monte Carlo analysis implemented in MATLAB is based on the algorithm described by equation (6.13). This algorithm is an exact simulation in the sense that the distribution of $X(t_1), ...X(t_n)$ produces precisely of the Ornstein-Uhlenbeck process at time $t_1, ..., t_n$ for the same initial value $X(0)$. Take the square of the above equation (6.13), we obtain the squared Ornstein-Uhlenbeck process U, i.e.

$$U(t_i) = (X(t_i))^2$$

Nevertheless, it is more instructive to visualise the trajectories of an Ornstein-Uhlenbeck process and a squared Ornstein-Uhlenbeck process. Fig.6.1 shows a realisation of an Ornstein-Uhlenbeck process (Top) and a realisation of a squared Ornstein-Uhlenbeck process (Bottom). (MATLAB implementation can be found in appendix C)

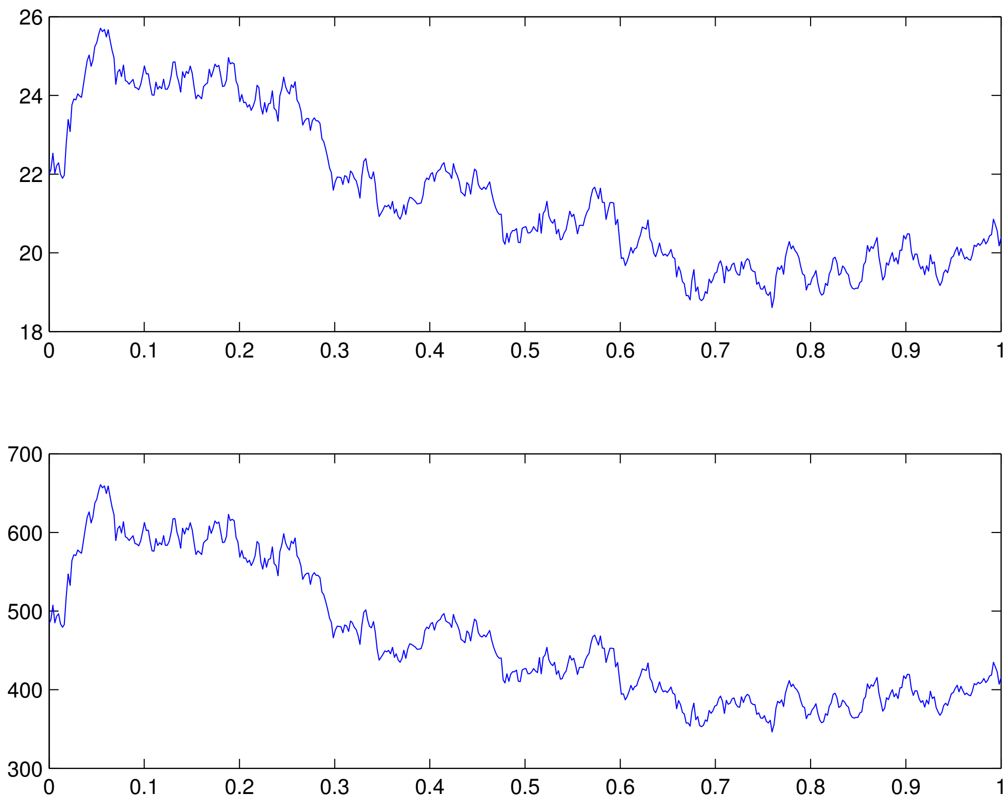


Figure 6.1: Trajectories of an Ornstein-Uhlenbeck process and a Square Ornstein-Uhlenbeck process

6.5 Simulation of the VG process

For a single dimensional VG process, there are several efficient methods including sequential sampling and bridge sampling techniques for constructing the sample paths. Sequential sampling based methods rely on the two different representations given in 4.1.1. That is, to write VG as either a subordinated Brownian motion or a difference of two independent gamma processes. Bridge sampling based methods sample the end of the path first, then fills in the rest of the path as needed. For the detail of simulating VG via bridge sampling, we refer readers to [31]. Here we refer to the paper [31] and describe the sequential sampling method in simulating a single dimensional VG process. There are three methods one can consider. The first two based on the two representation presented in 4.1.1 and are "exact" in the sense when the correct distribution is available. The algorithms are the following

Algorithm 1 (Simulating VG as time-changed Brownian Motion)**.**

INPUT: VG parameters θ, σ, ν; **time spacing** $\Delta t_1, ..., \Delta t_n$ **such that** $\sum_{i=1}^{N} \Delta t_i = T$

INITIALIZATION: Set $X(0) = 0$.

Loop from $i = 1$ **to** N**:**
1. Generate $\Delta G_i \sim \Gamma(\Delta t_i/\nu, \nu), Z_i \sim N(0,1)$
independently and independent of past r.v.s.
2. Return $X(t_i) = X(t_{i-1}) + \theta \Delta G_i + \sigma \sqrt{\Delta G_i} Z_i$.

Note: The letter G is reserved for the Gamma process in algorithm 1.

Algorithm 2 (Simulating VG as Difference of Gammas)**.**

INPUT: VG parameters θ, σ, ν; **time spacing** $\Delta t_1, ..., \Delta t_n$ **such that** $\sum_{i=1}^{N} \Delta t_i = T$

INITIALIZATION: Set $X(0) = 0$.

Loop from $i = 1$ **to** N**:**
1. Generate $\Delta\gamma_i^+ \sim \Gamma(\Delta t_i/\nu, \nu\mu_+), \Delta\gamma_i^- \sim \Gamma(\Delta t_i/\nu, \nu\mu_-)$
independently and independent of past r.v.s.
2. Return $X(t_i) = X(t_{i-1}) + \Delta\gamma_i^+ - \Delta\gamma_i^-$.

In this thesis, we have chosen to adopt algorithm 2 based on the difference of gamma parametrisation as described in equations (4.14) and (4.15). The main reason is that this algorithm appears to be simpler as it does not require the simulation of Brownian motion.

The third method is to approximate VG as a compound Poisson process. The main advantage of the third method is its generality. In other words, it can be used for any Lévy process. The algorithm can be found in [31].

6.6 Simulation of the VWAP option

6.6.1 Parameter Values

The parameter values [2] used in Monte Carlo simulation are

$$S_0 = 110; \quad r = 0; \quad K = 100; \quad \rho = 0; \quad r = 0$$

$$\sigma_{\text{VG}} = 0.1 \quad \sigma_{\text{BM}} = (0.1, 0.2, 0.3, 0.4, 0.5)$$

$$\nu = 0.1; \quad \theta = -0.14; \quad \sigma_{\text{OU}} = 5; \quad X(0) = 22; \quad T = 1$$

$$m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log\left(\frac{1}{1 - \theta\nu + \frac{(\sigma_{VG})^2}{2}\nu}\right) \Big/ \nu$$

$$N = 500 \quad M = 10^6$$

$$\Delta\gamma_i^- \sim \Gamma(\Delta t_i \frac{1}{\nu}, \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma_{\text{VG}}^2\nu}{2}} - \frac{\theta\nu}{2})$$

$$\Delta\gamma^+ \sim \Gamma(\Delta t_i \frac{1}{\nu}, \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma_{\text{VG}}^2\nu}{2}} + \frac{\theta\nu}{2})$$

6.6.2 Notation and Discretization

The VWAP option is priced over time period $[t_{\text{start}}, t_{\text{final}}]$ and set $T = t_{\text{start}} - t_{\text{final}}$. In this thesis we price from $t_{\text{start}} = 0$ to $t_{\text{final}} = 1$. The time period is discretized into $N = 500$ time intervals of equal with $\Delta t = \frac{T}{N}$ so that the simulation is performed at the times $t_{\text{start}} = t_0, t_1, ..., t_N = t_{\text{final}}$ where $t_i = i\Delta t, \Delta t_i = \Delta t, i = 1, ..., N$. The simulation was performed M times.

$S(t_i)$ is obtained by using the exact solution of $S(t)$ given by equation (5.1)

$$S(t_i) = S(t_{i-1})e^{m\Delta t + \sigma_{\text{BM}}\sqrt{\Delta t}(\rho\hat{Z}(t_i) + \sqrt{1-\rho^2}\tilde{Z}(t_i))}, \tag{6.14}$$

at each time step with $S_0 = S(0)$ and $Z(t_i)$ being drawn from $N(0,1)$ distribution[3].

Similarly, $X(t_i)$ is obtained by using the exact solution of $X(t)$ given by equation (6.8)

$$X(t_i) = X(t_{i-1})e^{-\lambda\Delta t} + a(1 - \lambda\Delta t) + \sqrt{\frac{\sigma_{\text{OU}}^2}{2\lambda}(1 - e^{-2\lambda\Delta t})}\hat{Z}(t_i)$$

Take the square of the above, we obtain the simulated square Ornstein-Uhblenbeck process.

6.6.3 The Algorithm

The algorithm in simulating the VWAP option combined all the algorithms discussed in this chapter, which can be viewed as a combined algorithm of simulating a European call option, an Ornstein-Uhlenbeck process and a geometric Lévy process with taking the jump component as a VG process. To obtain the price of a VWAP option we have to use Monte Carlo techniques again. To simulate the trajectories of the VWAP, we use the crude Monte Carlo method with 10^6 simulated paths to price the VWAP option. The algorithm is summarised as the following:

set Sum 1=Sum 2=Sum 3=Sum 4=0

[2] $\sigma_{\text{VG}}, \sigma_{\text{BM}}, \nu, \theta$ are, respectively, the skewness parameter, the diffusion parameter, the kurtosis parameter and the symmetry parameter

[3] MATLAB default normal random number generator **randn**() was used to produce $N(0,1)$ random variable.

for $j = 1, ..., M$ **do**
INITIALIZATION
for $i = 1, ..., N$ **do**
Generate $\Delta\gamma_i^+ \sim \Gamma(\Delta t_i/\nu, \nu\mu_+), \Delta\gamma_i^- \sim \Gamma(\Delta t_i/\nu, \nu\mu_-)$
Set $X(t_i) = 0$
Return $X(t_i) = X(t_{i-1}) + \Delta\gamma_i^+ - \Delta\gamma_i^-$
Generate a trajectory for stock price
Generate a trajectory for (trade)volume
Multiplying the stock price trajectory with volume trajectory $S(t)U(t)$
Sum 1=Sum 1+ volume
Sum 2=Sum 2+ price* volume
end for
$\bar{A_T} = \frac{\sum \text{price*volume}}{\sum \text{volume}}$
Sum 3=Sum 3+$\bar{A_T}$
$C_j = e^{-rT}(\bar{A_T} - K)^+$
Sum 4=Sum 4+C_j
end for
set $\hat{C}_M = (C_1 + ... + C_M)/M$

The MATLAB implementation can be found in appendix B.

6.6.4 Accuracy

In this chapter, we use the discrete-time definition of VWAP, so for each simulation j, the VWAP is found as

$$A_j = \frac{\sum_{i=0}^{N-1} S(t_i)U(t_i)}{\sum_{i=0}^{N-1} U(t_i)}$$

This is repeated $M = 10^6$ times in this thesis. Finally, we approximate the expectation of A_j by

$$\mathbb{E}(A) \approx \frac{1}{M}\sum_{j=1}^{M} A_j$$

and variance by

$$\text{Var}(A) = \frac{1}{M}\sum_{j=1}^{M} A_j^2 - (\frac{1}{M}\sum_{j=1}^{M} A_j)^2$$

and standard error by

$$s(A) = \sqrt{\frac{1}{M}\sum_{j=1}^{M} A_j^2 - (\frac{1}{M}\sum_{j=1}^{M} A_j)^2} \tag{6.15}$$

Hence the MC standard error of the first moment $\mathbb{E}(A_T)$ is calculated as in equation (6.15).

The MC standard error of the second moment $\mathbb{E}(A_T^2)$ is given by

$$s(A) = \sqrt{\frac{1}{M}\sum_{j=1}^{M} A_j^4 - (\frac{1}{M}\sum_{j=1}^{M} A_j^2)^2} \tag{6.16}$$

Chapter 7

Numerical Results

7.1 Results of the main model, the comparable results of the GBM model and the change of correlation ρ

In this chapter we will discuss the implementation based on the lognormal and the GIG approximations under the assumption that stock price evolves as a geometric Lévy process and volume evolves as a shifted squared Ornstein-Uhlenbeck process. The computation was carried out by running 10^6 crude Monte Carlo using MATLAB R2010b on a Macintosh Labtop with IntelCore I7. All calculations of the first moments were performed symbolically leading to exact expressions for equations (5.12) and (5.26). The subsequent multidimensional integrals were computed numerically using NIntegrate in Mathematica and are very fast. Monte Carlo simulation was used to estimate the third VWAP moment for pricing with the GIG. The Monte Carlo simulations were performed under MATLAB using $n = 10^6$ trajectories and 500 discretisation points over $[0, T]$. Our parameter choices give rise to

$$S_t = S_0 e^{rt+mt+X_t^{VG}+\sigma_{BM}\bar{W}_t^{(2)}}$$

for the stock price dynamics and

$$\begin{aligned} U_t &= X_t^2, \\ dX_t &= 2(22 - X_t)dt + 5d\bar{W}_t^{(1)}, \quad X_0 = 22 \end{aligned}$$

for the volume dynamics. The Brownian motion $\bar{W}_t^{(2)}$ under the stock price dynamics correlated with the Brownian motion $\bar{W}_t^{(1)}$ under the volume dynamics via the following equations

$$\begin{aligned} \bar{W}_t^{(1)} &= \widehat{W}_t \\ \bar{W}_t^{(2)} &= \rho\widehat{W}_t + \sqrt{1-\rho^2}\widetilde{W}_t \\ [\bar{W}^{(1)}, \bar{W}^{(2)}]_t &= \rho t, \quad \forall t \in [0, T] \end{aligned}$$

First, we set $T = 1, \rho = 0$ for traceability. Tables 7.1 and 7.2 display a range of computed moments and the corresponding simulated values in our geometric Lévy model. The accuracy of the first moment approximation is validated by the small Monte Carlo standard error and relative error [1]. Fig.7.1 displays call option prices for different strike values (K) and different stock price volatilities (σ_{BM}) for both lognormal and GIG approximations. Fig.7.2 shows call option prices for different strike values

[1] This is computed as $\frac{|\text{MC estimate} - \text{Analytical approximation}|}{\text{MC estimate}}$.

Table 7.1: Numerical values of $\mathbb{E}(A_T)$ and Monte Carlo simulation of $\mathbb{E}(A_T)$ for varying stock price volatility value σ_{BM} under the GL model, assuming $\rho = 0$.

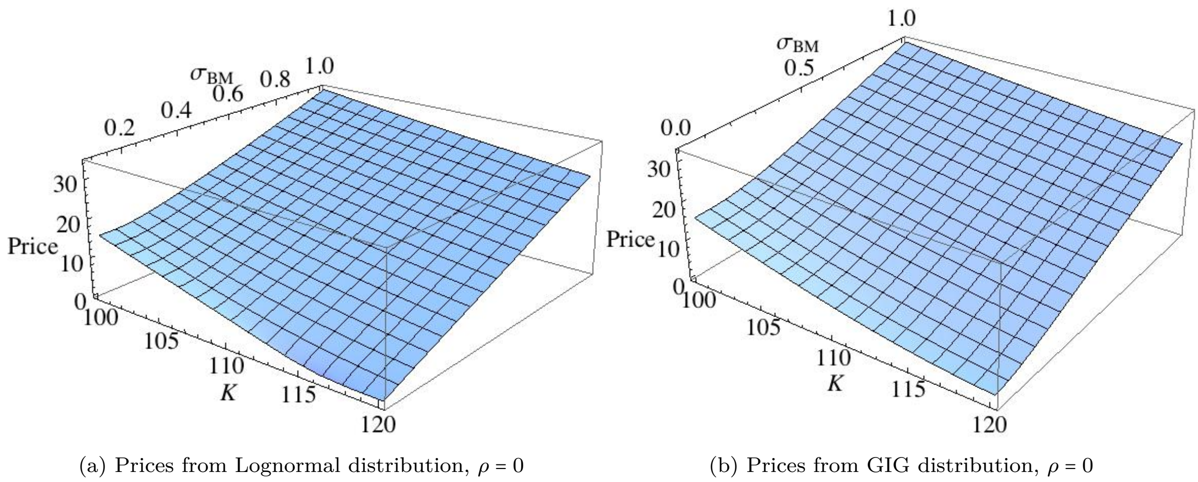
σ_{BM}	$\mathbb{E}(A_T)$ [a]	$\hat{\mathbb{E}}(A_T)$	MC std. error	Rel.error (%)
0.1	115.68	115.680	0.0099	0.0009
0.2	115.68	115.679	0.0154	0.0017
0.3	115.68	115.678	0.0203	0.0017
0.4	115.68	115.677	0.0286	0.0026
0.5	115.68	115.676	0.0358	0.0035

[a] Note that take $\rho = 0$, drift, symmetry,kurtosis,skewness parameters are held constant as $\mu = 0.1, \sigma = 0.1, \theta = 0.14, v = 0.1$ and $\sigma_{VG} = 0.1$
σ_{BM} does not enter into the computation of $\mathbb{E}(A_T)$, which leads to unchanging values for this column

Table 7.2: Numerical values of $\mathbb{E}(A_T^2)$ and Monte Carlo simulation of $\mathbb{E}(A_T^2)$ for varying stock price volatility value σ_{BM} under the GL model, assuming $\rho = 0$.

σ_{BM}	$\mathbb{E}(A_T^2)$	$\hat{\mathbb{E}}(A_T^2)$	MC std. error	Rel.error (%)
0.1	13392.73	13480.69	2.3199	0.6525
0.2	13531.17	13619.98	3.695	0.6521
0.3	13766.66	13760.3	5.4215	0.6528
0.4	14106.67	14199.7	7.5000	0.6552
0.5	14562.29	14659.06	10.058	0.6602

(K) and different stock price volatilities (σ_{BM}) under the Monte Carlo Simulation using $n = 1,000,000$ trajectories and 500 discretisation points over one year. Fig.7.4 shows computed prices arising from different methods with σ_{BM} ranging over $[0.2, 0.5]$. Relative error plots comparing approximated prices with simulated counterparts are presented in Fig.7.3.



(a) Prices from Lognormal distribution, $\rho = 0$ (b) Prices from GIG distribution, $\rho = 0$

Figure 7.1: Call option prices for different strike values K and stock price volatility $\sigma_{\text{BM}}, \rho = 0$ (GL).

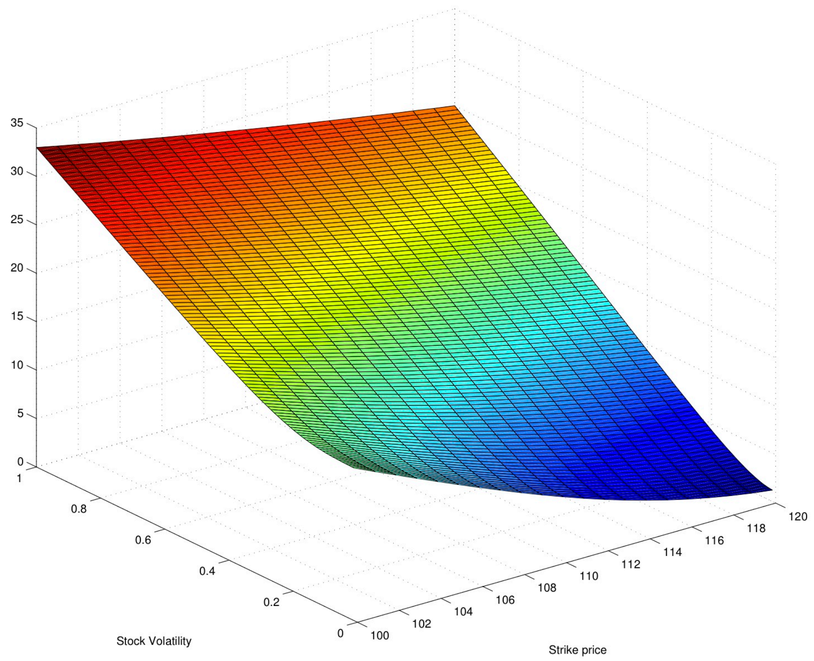
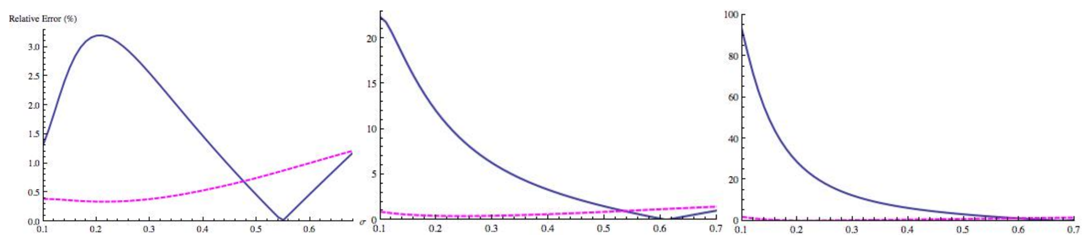


Figure 7.2: Simulated Call option prices (GL) for different strike price and different stock price volatility $\sigma_{\rm BM}, \rho = 0$.

For $K = 100$, the relative error of the lognormal approximation increases for $\sigma_{BM} < 0.2$, then decreases in the volatility range $[0.2, 0.5]$ with a tendency to increase. The relative error of the GIG approximation stays beneath 1.26% over the entire range of volatilities.

For $K = 110$, the relative error of the lognormal approximation exceeds 10.44% for small $\sigma_{BM} < 0.22$, the relative error of the GIG approximation appears to be quite stable and quite small. The relative error of the GIG approximation stays beneath 1.46% over the entire range of volatilities.

For $K = 120$, the lognormal approximation deteriorates even more while the GIG approximation works exceptionally well.



(a) $K = 100, \rho = 0$ (b) $K = 110, \rho = 0$ (c) $K = 120, \rho = 0$

Figure 7.3: Relative error (GL, $\rho = 0$) of option prices as a function of $\sigma_{\rm BM}$. The blue line represents Lognormal error and the pink dashed line is the GIG error.

Fig.7.4 shows how well the two approximation methods work. The GIG prices appear to be very close to the Monte Carlo benchmark prices while the lognormal approximation tends to underprice toward lower volatilities range.

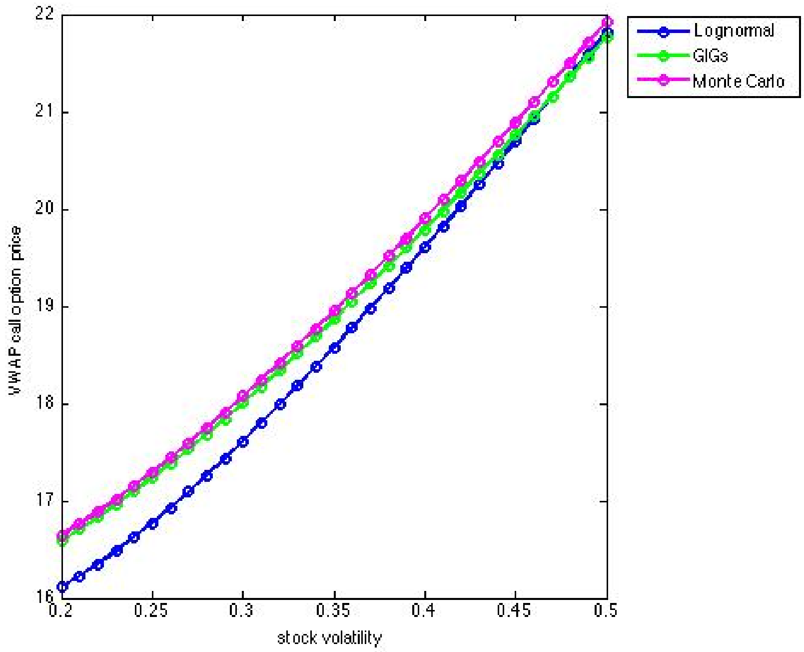


Figure 7.4: Call option prices (GL) for $K = 100$ and different stock price volatility $\sigma_{\mathrm{BM}}, \rho = 0$.

Next, we compare against our results with the classical geometric Brownian model. The range of computed moments and the corresponding simulated values are presented in Tables 7.3 and 7.4. The accuracy of the first moment approximation is also validated by the small Monte Carlo standard error and relative error. Fig.7.5 displays call option prices for different strike values (K) and different stock price volatilities (σ_{BM}) for both lognormal and GIG approximation. Fig.7.6 shows call option prices for different strike and different stock price volatilities (σ_{BM}) under Monte Carlo Simulation using $n = 1,000,000$ trajectories and 500 discretisation points over one year. Fig.7.8 shows computed prices arising from different methods with σ_{BM} ranging over $[0.2, 0.5]$. Relative error plots comparing approximated prices with simulated counterparts are presented in Fig.7.7.

The parameters for asset price and volume were chosen to be as similar as possible to those in papers Novikov et al. [62] and Stace [80] and it is worth mentioning that the computed parameter values are quite similar to their results. The parameter choices give rise to

$$dS_t = (0.1)S_t dt + \sigma_{\mathrm{BM}} S_t d\bar{W}_t^{(2)}, S_0 = 110$$

for the stock price dynamics and

$$\begin{aligned} U_t &= X_t^2, \\ dX_t &= 2(22 - X_t)dt + 5d\bar{W}_t^{(1)}, \quad X_0 = 22 \end{aligned}$$

for the volume dynamics. The Brownian motion $\bar{W}_t^{(2)}$ under the stock price dynamics correlated with the Brownian motion $\bar{W}_t^{(1)}$ under the volume dynamics via the following equations

$$\begin{aligned} \bar{W}_t^{(1)} &= \widehat{W}_t \\ \bar{W}_t^{(2)} &= \rho\widehat{W}_t + \sqrt{1-\rho^2}\widetilde{W}_t \\ [\bar{W}^{(1)}, \bar{W}^{(2)}]_t &= \rho t, \quad \forall t \in [0, T] \end{aligned}$$

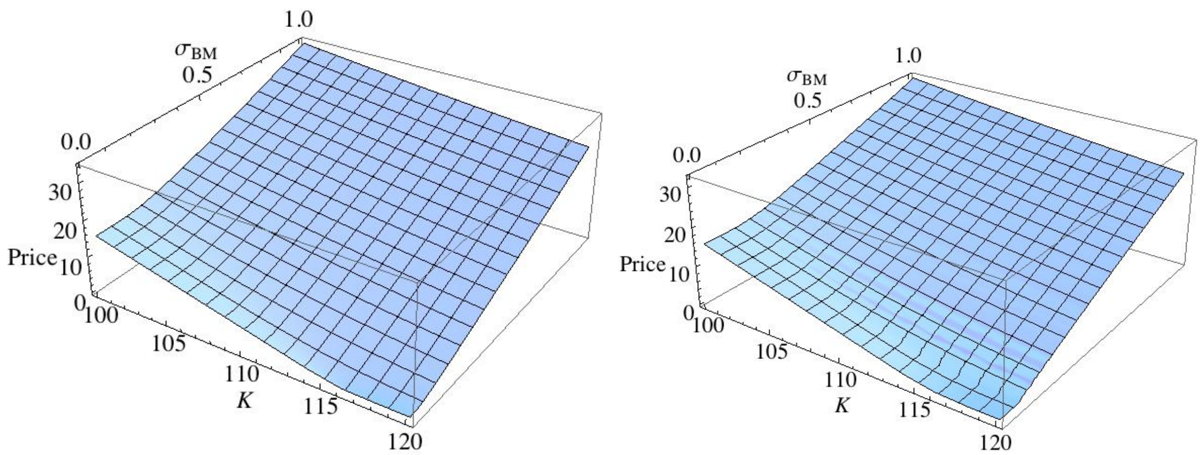
As usual, we set $T = 1, \rho = 0$ for traceability.

Table 7.3: Numerical values of $\mathbb{E}(A_T)$ and Monte Carlo simulation of $\mathbb{E}(A_T)$ for varying stock price volatility value σ_{BM} under the GBM model, assuming $\rho = 0$

σ_{BM}	$\mathbb{E}(A_T)$	$\hat{\mathbb{E}}(A_T)$	MC std. error	Rel.error (%)
0.1	115.68	115.4507	0.0068	0.2094
0.2	115.68	115.4428	0.0136	0.2162
0.3	115.68	115.4349	0.0205	0.2229
0.4	115.68	115.4271	0.0276	0.2295
0.5	115.68	115.4195	0.0349	0.2395

Table 7.4: Numerical values of $\mathbb{E}(A_T^2)$ and Monte Carlo simulation of $\mathbb{E}(A_T^2)$ for varying stock price volatility value σ_{BM} under the GBM model, assuming $\rho = 0$.

σ_{BM}	$\mathbb{E}(A_T^2)$	$\hat{\mathbb{E}}(A_T^2)$	MC std. error	Rel.error (%)
0.1	13427.92	13374.76	1.5758	0.3975
0.2	13566.90	13511.73	3.2294	0.4083
0.3	13803.32	13746.04	5.0545	0.4167
0.4	14144.68	14085.19	7.1647	0.4224
0.5	14602.11	14540.34	9.7154	0.4249



(a) Prices from Lognormal distribution, $\rho = 0$.

(b) Prices from GIG distribution, $\rho = 0$.

Figure 7.5: Call option prices for different strike values K and stock price volatility $\sigma_{\text{BM}}, \rho = 0$ (GBM).

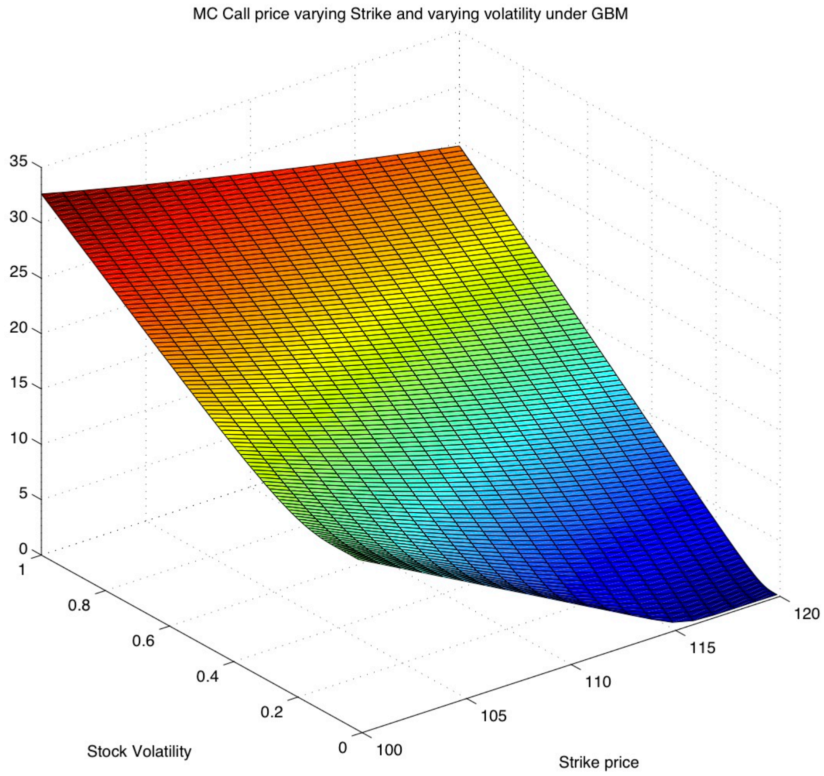
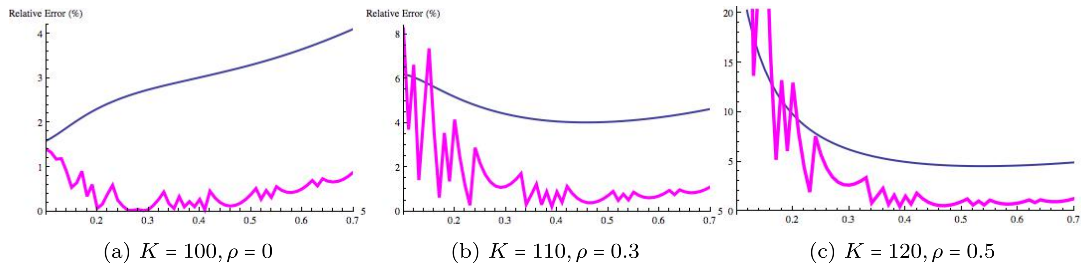


Figure 7.6: Simulated Call option prices (GBM) for different strike price and different stock price volatility $\sigma_{\mathrm{BM}}, \rho = 0$.

For $K = 100$ it can be seen that the relative error of the lognormal approximation stays within 3% in the volatility range $[0.2, 0.39]$ with a tendency to increase. The relative error for GIG approximation stays beneath 0.6% over the entire range of volatilities.

For $K = 110$, the relative error for the lognormal approximation presents a stable decreasing trend while the relative error for the GIG approximation presents a zigzag pattern for $\sigma_{\mathrm{BM}} < 0.2$, this indicates the GIG approximation appears to be unstable for numerical calculation in small volatilities range. These findings for VWAP options do conform with a classical market observation for Asian options in some sense, namely, for a small volatility of underlying process and near at-the-money options the log-normal approximation is not too bad [46; 62]. For $K = 120$, from Fig.7.7c we see that the relative error of both approximations reaching the maximum magnitude. Also, as σ_{BM} increases, the relative error tapers off to zero.



(a) $K = 100, \rho = 0$ (b) $K = 110, \rho = 0.3$ (c) $K = 120, \rho = 0.5$

Figure 7.7: Relative error (GBM, $\rho = 0$) of option prices as a function of σ_{BM}. The blue line represents Lognormal error and the pink line is the GIG error.

Fig.7.8 shows how well the two approximation methods work. The GIG approximation appears to be very close to the Monte Carlo benchmark while the lognormal approximation tends to overprice toward higher volatilities range.

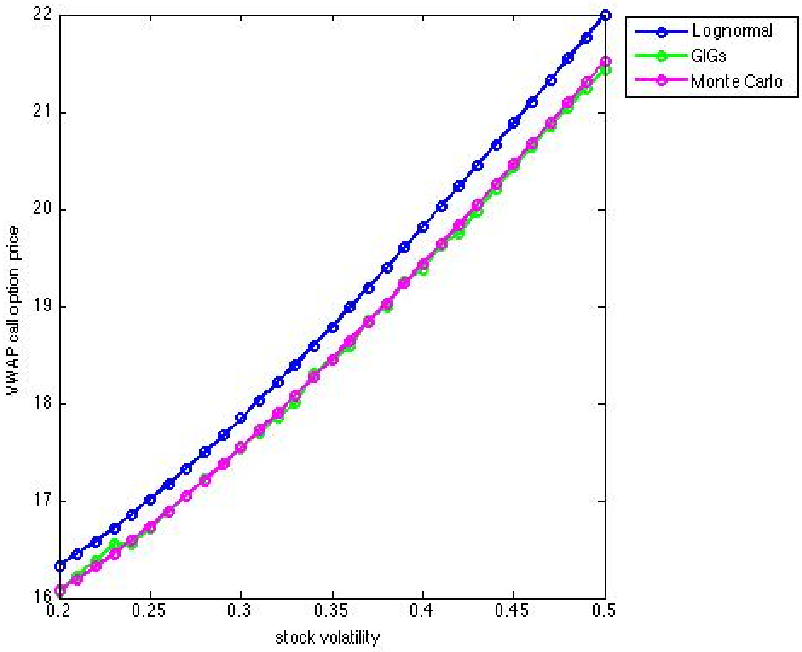


Figure 7.8: Call option prices (GBM) for $K = 100$ and different stock price volatility $\sigma_{\text{BM}}, \rho = 0$.

Third, we examine the effect of the change of correlation level ($\rho = 0.3$, $\rho = 0.5$) between the driving Brownian motions ($\bar{W}_t^{(1)}$ and $\bar{W}_t^{(2)}$) on the moments of VWAP and the price of the option. As indicated in table 7.5, 7.6, 7.7 and 7.8, the relative error for both moments appears to be bigger than the original scenario. For the GIG approximation, the relative error in Fig.7.14 exhibits some zigzag pattern in low volatilities range $\sigma_{\text{BM}} < 0.175$. For the lognormal approximation, the accuracy becomes highly unreliable. Fig.7.11 and 7.12 indicate that the lognormal prices are much lower than the Monte Carlo benchmark prices over the entire range of volatilities $[0.2, 0.5]$.

Table 7.5: Numerical values of $\mathbb{E}(A_T)$ and Monte Carlo simulation of $\mathbb{E}(A_T)$ for varying stock price volatility value σ_{BM} under the GL model, assume $\rho = 0.3$.

σ_{BM}	$\mathbb{E}(A_T)$	$\hat{\mathbb{E}}(A_T)$	MC std. error	Rel.error (%)
0.1	115.68	116.23	0.0100	4.7
0.2	115.68	116.77	0.0156	0.935
0.3	115.68	117.31	0.0222	1.388
0.4	115.68	117.84	0.029328	1.833
0.5	115.68	118.37	0.0369	2.270

Table 7.6: Numerical values of $\mathbb{E}(A_T)$ and Monte Carlo simulation of $\mathbb{E}(A_T)$ for varying stock price volatility value σ_{BM} under the GL model, assume $\rho = 0.5$.

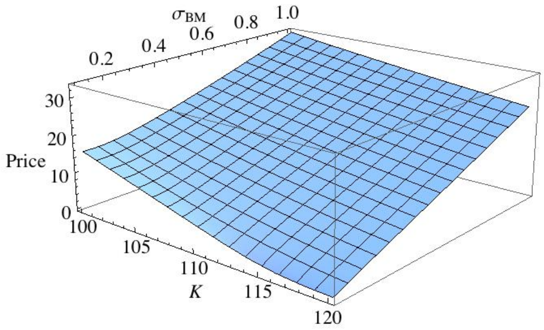
σ_{BM}	$\mathbb{E}(A_T)$ [a]	$\hat{\mathbb{E}}(A_T)$	MC std. error	Rel.error (%)
0.1	115.68	116.18	0.0100	4.7
0.2	115.68	116.68	0.0156	0.857
0.3	115.68	117.17	0.022201	1.271
0.4	115.68	117.66	0.029373	1.683
0.5	115.68	118.15	0.03678	2.091

Table 7.7: Numerical values of $\mathbb{E}(A_T^2)$ and Monte Carlo simulation of $\mathbb{E}(A_T^2)$ for varying stock price volatility value σ_{BM} under the GL model, assuming $\rho = 0.3$.

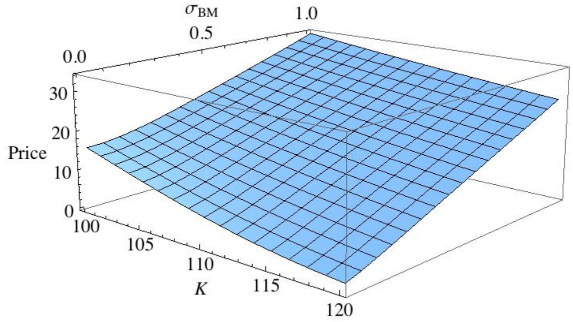
σ_{BM}	$\mathbb{E}(A_T^2)$	$\hat{\mathbb{E}}(A_T^2)$	MC std. error	Rel.error (%)
0.1	13388.61	13610.97	2.3603	1.63
0.2	13514.44	13881.2	3.8019	2.64
0.3	13728.1	14255.77	5.6462	3.70
0.4	14035.72	14746.56	7.9126	4.82
0.5	14446.38	15369.81	10.7554	6.01

Table 7.8: Numerical values of $\mathbb{E}(A_T^2)$ and Monte Carlo simulation of $\mathbb{E}(A_T^2)$ for varying stock price volatility value σ_{BM} under the GL model, assuming $\rho = 0.5$.

σ_{BM}	$\mathbb{E}(A_T^2)$	$\hat{\mathbb{E}}(A_T^2)$	MC std. error	Rel.error (%)
0.1	13381.29	13480.69	2.3199	1.59
0.2	13484.78	13619.98	3.695	2.69
0.3	13659.94	14221.3	5.6295	3.95
0.4	13910.88	14701	7.8855	5.37
0.5	14243.64	15311	10.718	6.97

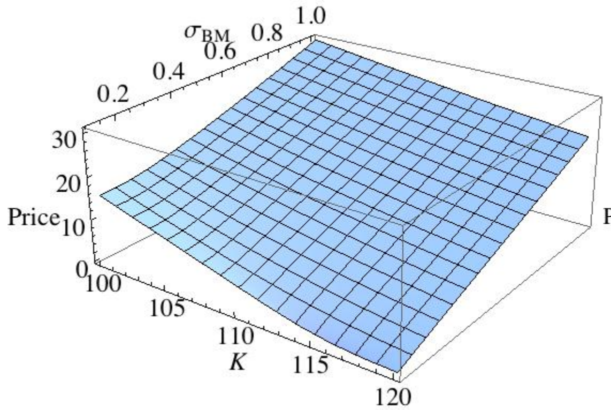


(a) Prices from Lognormal distribution, $\rho = 0.3$

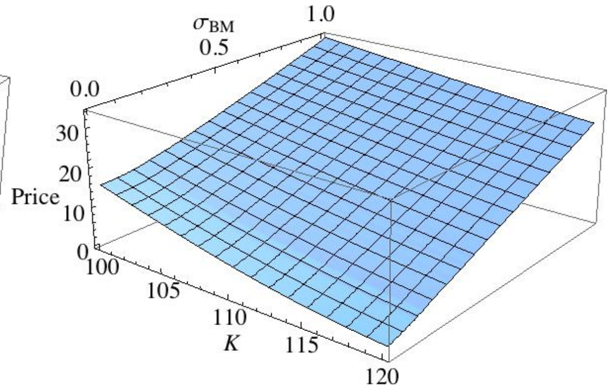


(b) Prices from GIG distribution, $\rho = 0.3$

Figure 7.9: Call option prices for different strike values K and stock price volatility $\sigma_{\text{BM}}, \rho = 0.3$ (GL).



(a) Prices from Lognormal distribution, $\rho = 0.5$



(b) Prices from GIG distribution, $\rho = 0.5$

Figure 7.10: Call option prices for different strike values K and stock price volatility $\sigma_{\text{BM}}, \rho = 0.5$ (GL).

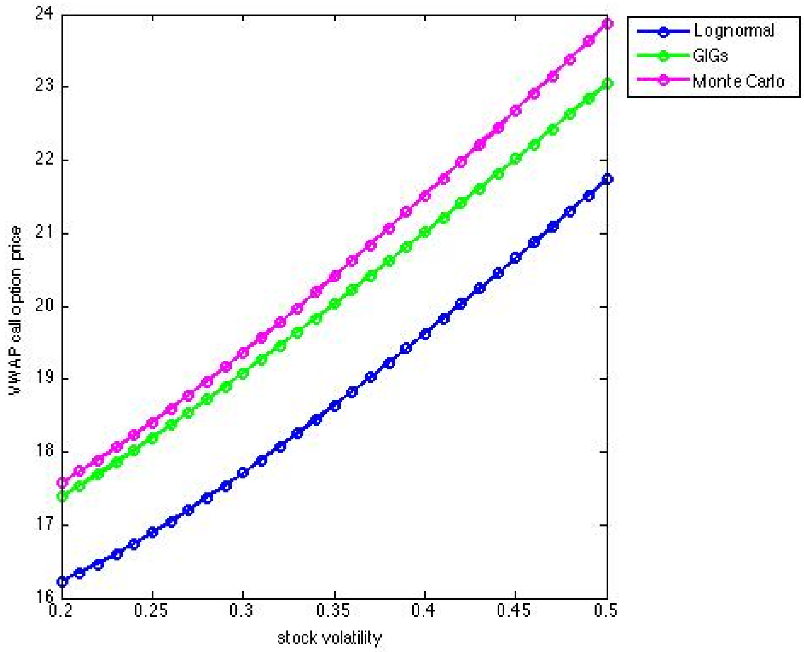


Figure 7.11: Call option prices (GL) for $K = 100$ and different stock price volatility σ_{BM}, $\rho = 0.3$.

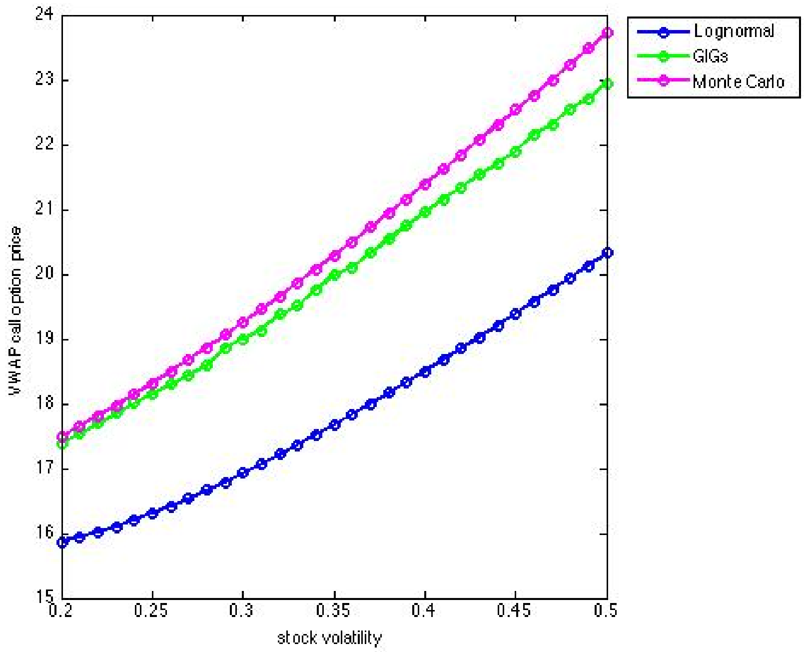
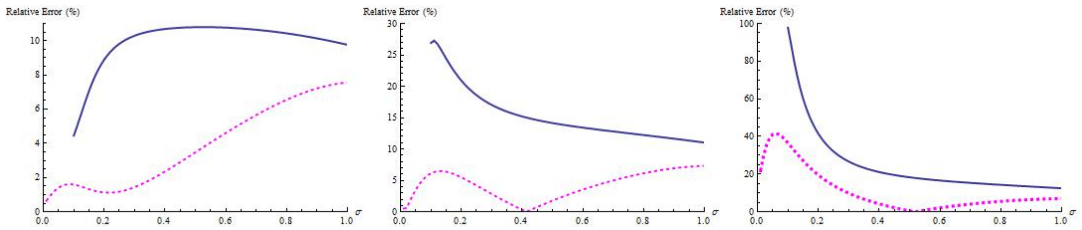
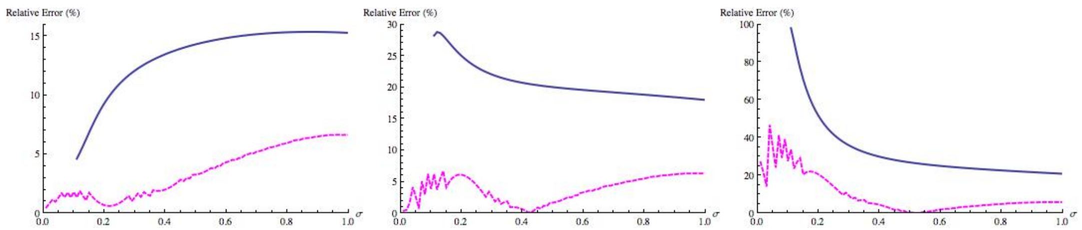


Figure 7.12: Call option prices (GL) for $K = 100$ and different stock price volatility σ_{BM}, $\rho = 0.5$.



(a) $K = 100, \rho = 0.3$ (b) $K = 110, \rho = 0.3$ (c) $K = 120, \rho = 0.3$

Figure 7.13: Relative error (GL, $\rho = 0.3$) of option prices as a function of σ_{BM}. The blue line represents Lognormal error and the pink dashed line is the GIG error.



(a) $K = 100, \rho = 0.5$ (b) $K = 110, \rho = 0.5$ (c) $K = 120, \rho = 0.5$

Figure 7.14: Relative error (GL, $\rho = 0.5$) of option prices as a function of σ_{BM}. The blue line represents Lognormal error and the pink dashed line is the GIG error.

Three main points are illustrated based on the observations across models and correlation levels.

First, for the strike level where options are at-the-money, i.e. $K = 110$, the relative error of lognormal approximation is found to be quite small under the GBM model but is found to be quite big in our GL model. This indicates that the classical market observation for Asian options that the lognormal outperforms others does not hold under our GL model. Indeed, the GIG approximation seems to be more suitable for pricing small volatility and near at-the-money options based on the small size and smooth pattern of the relative error. Overall, it appears that the lognormal approximation is unsuitable for VWAP whereas the underlying stock price dynamics is described by a geometric Lévy process. This is different from the classical observation for Asian options (where the underlying asset dynamics follows a geometric Brownian motion) that lognormal approximations outperform others (See papers [46; 62]). When the driving Brownian motions are correlated, both approximations deteriorate (See Fig.7.13 and Fig.7.14). Nevertheless, the accuracy of GIG approximation remains considerably reliable, despite some zigzag pattern at low volatility range, i.e. $\sigma_{\text{BM}} < 0.175$.

Second, the zigzag pattern that is usually found in the relative error plot of the GIG approximation towards small volatilities range in the GBM model seems to disappear in our GL model. The GIG approximation seems to be quite stable and provides more reliable accuracy than the lognormal approximation.

Third, as the correlation level changes, the capability of both approximations deteriorates. The relative error of the GIG approximation is still considerably small, but the zigzag pattern in low volatilities range may suggest some instability of the numerical calculations. On the other hand, the size of the relative error suggests the lognormal approximation is more vulnerable than the GIG approximation in response to the change of correlation level. The option prices comparison plots indicate the lognormal approximation significantly underprice the VWAP option. This observation relates to the phenomenon of volatility skew and smiled mentioned earlier in 1.1.1 and does conform with the classical observation that out-of-the-money options tend to be underpriced in model with continuous trajectories [83].

Overall, it appears that the lognormal approximation is unsuitable for VWAP whereas the underlying stock price dynamics is described by a geometric Lévy process, which is different from the classical observation for Asian options that lognormal approximation outperforms. Despite some zigzag pattern at low volatility range, the accuracy of GIG approximation remains considerably reliable.

Table 7.9 summarises the above discussion and provides a performance comparison of the two approximation methods under the two pricing setups for different strike values (K) and stock price volatilities (σ_{BM}).

Table 7.9: Relative error comparison across methods and models

	$\sigma_{\text{BM}} \in (0, 0.2)$	$\sigma_{\text{BM}} \in [0.2, 0.5)$	$\sigma_{\text{BM}} \in [0.5, 1]$
GL			
$K = 100$	GIG	Lognormal	GIG
$K = 110$	GIG	GIG	Lognormal
$K = 120$	GIG	GIG	GIG
GBM			
$K = 100$	Both are fine	GIG	GIG
$K = 110$	Lognormal. Instability over GIG	GIG	GIG
$K = 120$	Lognormal. Instability over GIG	GIG	GIG

7.2 Comment on MC implementation

The relative error plots suggest that the GIG approximation appears to be more successful in approximating accurate prices than the lognormal approximation. However, it shall be stressed that the crude MC algorithm implemented in MATLAB requires significant computational effort: The CPU time to obtain 100 option prices against 20 strike prices is about 13 hours in our setup while only 8 hours are needed in the geometric Brownian motion setup, assuming a MC simulation with 10^6 trials is used and the parameter values are chosen as close as possible to those in paper [62]. Despite the fact that this time could be significant reduced if a lower-level programming language (e.g. C++) could be used, some possible reasons for this disappointment are worth mentioning. *First*, despite the simplicity, a well-known drawback of the crude Monte Carlo methods is their slow convergence, which can make the estimation process very time consuming if a more precise estimator is required. *Second*, along with the mentioned advantages, a difficulty in using the VG process and the more general Lévy process is that they require more advanced stochastic analysis. In the case of valuing a path-dependent option, a closed-form solution cannot be found [38]. Various techniques are suggested to improve the efficiency, such as stratified sampling, bridge sampling and Quasi-Monte Carlo (QMC) methods have been developed. In the paper [72], a time-changed Brownian motion gamma bridge [2] is introduced, along with stratified sampling and QMC to reduce variance. The paper by Avramidis and L'Ecuyer (2006) [3] introduces a difference of gamma bridge [3] combined with QMC. An excellent tutorial overview of VG and Monte Carlo can be found in Fu (2000) [31].

[2] Bridge sampling for the time-changed Brownian motion representation

[3] Bridge sampling combined with randomised QMC for the difference-of-gamma representation

Chapter 8

Conclusions and perspective

8.1 Overall Conclusion

This thesis seeks to extend the previous work [62] on the pricing of VWAP options [1], by allowing the stock price to be modelled with a Lévy process. The semi-analytical approach developed by the paper Novikov et al. [62] was studied and explored. Explicit formulae for the moments of VWAP were found under the geometric Lévy process setup.

Having thoroughly studied the Lévy process (the underlying stock price process of the VWAP) in Chapters 2 and 4, analytical expressions for the first two moments of VWAP were presented and numerically implemented based on the semi-analytical method developed by Novikov et al. The approximation for the VWAP option prices were obtained by matching the first three (two) moments of the VWAP to a GIG (lognormal) distributions.

A Monte Carlo analysis was then performed to benchmark the numerical results for the moments of VWAP and call option prices. It was found the GIG approximation provides a fairly accurate approximation when the underlying stock price is assumed to evolve as a geometric Lévy process. On the other hand, the lognormal approximation that usually works well in pricing low volatilities and near at-the-money Asian option under the classical geometric Brownian motion model appears to be unreliable. It tends to underprice the VWAP option at all level of volatilities and strike values under the geometric Lévy model. Overall, under the geometric Lévy pricing model, the GIG distribution was found to be successful in approximating accurate VWAP option prices for all level of volatilities and strike values. The capability of the GIG approximation only deteriorates slightly as correlation is imposed between price and volume.

Hence, to approximate the price of VWAP options via the moment matching technique under a Lévy based model, the GIG approximation is preferred.

8.2 Recommendation for Future Research

If one were to conduct further research from the viewpoint of the above conclusion, one possible extension to this work is to match higher order moments by using a better numerical method such as the Gram-Charlie expansion.

Also, to achieve a higher level of accuracy when running MC simulations, a Quasi Monte Carlo method that uses low discrepancy sequences (LDS) could be considered. An example of LDS is the Sobol sequences. Sobol sequences possess all the statistical properties of random sequences, they often

[1] In the original work, stock price is assumed to evolve as a GBM

lead to a faster convergence to the desired simulated values when normal distribution is sampled. These sequences return normal pseudorandom variates without any random errors.
To utilise LDS, one may consider some advanced Monte Carlo algorithms, such as

- A time-changed Brownian motion gamma bridge in conjunction with stratified sampling and Quasi Monte Carlo (QMC);
- A difference of gamma bridge in conjunction with Randomised Quasi Monte Carlo (RQMC).

In this thesis, we considered only a one-dimensional problem. Hence, another possible extension of this thesis is to extend the pricing method to VWAP options which depends on more than one underlying asset. In particular, it is useful to develop a multidimensional (Lévy) model that takes into account the dependence between various assets and dependence between prices and volume.

Lastly, from a practical point of view, the next step is to develop an effective hedging strategy.

Appendix A

Mathematical Preliminaries and the calculation of covariance function

A.1 Mathematical Preliminaries

This section summarises some Mathematical results and definitions that have been used in this thesis. These results are thoroughly described by various textbooks in Mathematical Finance: Cont & Tankov (2004), Klebaner (2005), Glasserman (2003), Protter (2003 or 2005) and Musiela & Rutkowski (2005). As well as Lecture notes in Analysis, Stochastic Process & Stochastic Analysis.

Definition A.1 (σ-algebra)**.** Given the non empty set Ω, a collection of subsets of $\Omega, \mathcal{F}$, is called a σ-algebra if

- The empty set, $\varnothing$, is in $\mathcal{F}$. i.e. $\varnothing \in \mathcal{F}$,
- If a set, E, is an element of $\mathcal{F}$, then the complement of E, E^C, is also in $\mathcal{F}$,
- If a sequence of sets $E_1, E_2, ...$ belongs to $\mathcal{F}$. Then so does their union, i.e. $\cup_{i=1}^{\infty} E_i \in \mathcal{F}$,
- The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition A.2 (Probability Measure)**.** A probability measure, $\mathbb{P}$, on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \mapsto [0,1]$ such that

- $\mathbb{P}(\varnothing) = 0, \mathbb{P}(\Omega) = 1$
- if $E_1, E_2, ... \in \mathcal{F}$ and $\{E_i\}_{i=1}^{\infty}$ is disjoint then $\mathbb{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$

Definition A.3 (Equivalence Probability Measure)**.** Given a non empty set Ω and a σ-algelbra, $\mathcal{F}$, of Ω. Then two different probability measure $\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent if they agree on which sets in $\mathcal{F}$ have probability 0.

Definition A.4 (Probability Space)**.** The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Definition A.5. If $\Omega = \mathbb{R}^d$, then the Borel σ-algebra is defined as the σ-algebra generated by the open set on $\mathbb{R}^d$

Definition A.6 ($\mathcal{F}$-measurable)**.** Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the the function $f : \Omega \to \mathbb{R}^d$ is called $\mathcal{F}$-measurable if

$$f^{-1}(A) := \{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$$

for all open sets $A \in \mathbb{R}^d$.

Definition A.7 (Random Variable)**.** A random variable X is $\mathcal{F}$-measurable function

$$X : \Omega \longrightarrow \mathbb{R}^d$$

Definition A.8 (Cumulative Distribution Function)**.** The cumulative distribution function of the random variable X is the function F given by

$$F(x) = \mathbb{P}(X \le x),\ x \in \mathbb{R}$$

Definition A.9 (Probability Density Function)**.** In case where distribution function F in A.8 is differentiable, the probability density function $f(x)$ of $F(x)$ is

$$f(x) = \frac{dF(x)}{dx}$$

Definition A.10 (Gaussian density)**.** The probability of the Gaussian distribution with mean μ and variance σ^2 is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)}{2\sigma^2}\right),\ \forall\, x \in \mathbb{R}$$

Shorthand notation $N(\mu, \sigma^2)$ is often used to denote this distribution. The special case when $\mu = 0$ and $\sigma^2 = 1$, i.e. $N(0,1)$, is called the standard Gaussian distribution or standard normal distribution.

Definition A.11 (Expectation)**.** Given the random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty$, then the expectation of X is

$$\mathbb{E}X = \int_\Omega X(\omega) d\mathbb{P}(\omega)$$

In one dimension, the expectation of the continuous random variable X with probability density function $f(x)$ is

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x) dx$$

Definition A.12 (Conditional Expectation)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is either nonnegative or integrable. The conditional expectation of X given $\mathcal{G}$, denoted by $\mathbb{E}(X|\mathcal{G})$, is any random variable that satisfies

- Measurability: $\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable, and
- Partial averaging:

$$\int_A \mathbb{E}(X|\mathcal{G})(\omega) d\mathbb{P}(\omega) = \int_A X(\omega) dP(\omega),\ \forall\, A \in \mathcal{G}$$

Proposition A.1 (General Properties of Condition Expectation)**.** *Given the random variable ξ and ζ on $(\Omega, \mathcal{F}, \mathbb{P})$, then the following properties hold*

1. *Linearity:* $\mathbb{E}(a\xi + b\zeta) = a\mathbb{E}(\xi) + b\mathbb{E}(\zeta)\ \forall\ a, b \in \mathbb{R}$,
2. *Tower Law: If $\mathcal{H} \subset \mathcal{F}$, then* $\mathbb{E}(\mathbb{E}(\xi|\mathcal{F})|\mathcal{H}) = \mathbb{E}(\xi|\mathcal{H})$
3. *Iterated Expectation (Special Case of Tower Law):* $\mathbb{E}(\mathbb{E}(\xi|\mathcal{F})) = \mathbb{E}\xi$
4. $\mathbb{E}(\xi\zeta|\mathcal{G}) = \xi\mathbb{E}(\zeta|\mathcal{G})$ *if ξ is $\mathcal{G}$-measurable (taking out what is known)*
5. *Comparison: If $\xi \le \zeta$, i,e, $\mathbb{P}(\xi \le \zeta) = 1$, then*

$$\mathbb{E}(\xi) \le \mathbb{E}(\zeta)$$

.

6. Jensen's inequality : if φ is convex, real-valued function defined on $\mathbb{R}$, then

$$\varphi(\mathbb{E}\xi) \leq \mathbb{E}(\varphi(\xi)).$$

Definition A.13 (Variance)**.** Given the random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the variance of X is given by

$$Var(X) = \mathbb{E}((X - \mathbb{E}X)^2)$$

It is trivial to show

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

Definition A.14 (Covariance)**.** Given the random variables X, Y on the probability space X, Y on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the covariance of X and Y is given by

$$Cov(X,Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y)$$

or, in a more convenient form

$$Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$$

Definition A.15 (Correlation Coefficient)**.** Let X, Y be random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Further, assume that $Var(X) > 0$ and $Var(Y) > 0$. Then the correlation coefficient of X and Y is

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

The is often call correlation.

Definition A.16 (Standard Deviation)**.** The standard deviation of random variable X is given by

$$\sigma(X) = \sqrt{Var(X)}$$

The expectation is also called the first moment or mean, and the variance as the second moment. Moments can be defined for any order.

Definition A.17 (i^{th} Moment)**.** . The i^{th} moment of the random variable X is given by

$$m_i = \mathbb{E}(X^i)$$

Definition A.18 (Independent)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{F}$ and $\mathcal{H}$ be sub-σ-algebras. If $A \in \mathcal{G}$ and $B \in \mathcal{H}$, we have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

then we say that these two σ-algebras are independent.

Definition A.19 (Almost Sure Convergence)**.** A sequence of random variable $(X_n)_{n\geq 1}$ converges almost surely to a random variable X if

$$B = \{\omega : \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\} \text{ has } \mathbb{P}(B) = 0.$$

or

$$\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$$

Almost sure convergence is often abbreviated by

$$\lim_{n\to\infty} X_n = X \text{ a.s.}$$

Definition A.20 ($L^{\mathbb{P}}$-Space)**.** Given the random variable $X : \omega \to \mathbb{R}^d$, then we define the L^2-norm of X, denoted by $\|X\|_2$, by

$$\|X\|_2 = \sqrt{\int_\Omega |X(\omega)|^2 d\mathbb{P}(\omega)}$$

The $L^2(\mathbb{P})$-space is given by

$$L^2(\mathbb{P}) = L^2(\Omega) = \{X : \Omega \to \mathbb{R}^d; \|X\|_2 < \infty\}$$

Definition A.21 (Convergence in L^p)**.** A sequence of random variables $(X_n)_{n\geq 1}$ converges in L^p to X (where $1 \leq p < \infty$) if $|X_n|, X$ are in L^p and

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0$$

Definition A.22 (Converge in Distribution)**.** The sequence of distributions of r.v.'s $X_1, X_2, ...$ converges to the distribution of r.v. X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for any x from the set of continuity points of $F_X(x)$.
Notations: $X_n d \xrightarrow{d} X$

Definition A.23 (Weak Convergence)**.** The sequence $X_1, X_2, ...$ converges weakly to X if

$$\lim_{n\to\infty} \mathbb{E}(F(X_n)) = \mathbb{E}(f(X))$$

for every bounded continuous function $f(x)$ Notations: $X_n d \xrightarrow{d} X$

Theorem A.2 (Weak Law of Large Numbers (WLLN))**.** *Let $X_1, X_2, ..., X_n$ be independent random variables with finite expected value $\mathbb{E}(X_i) = \mu$ and finite variance $V(X_i) = \sigma^2$. Let $S_n = \sum_{i=1}^n X_i$. Then for any $\epsilon > 0$*

$$\mathbb{P}\left(|\frac{S_n}{n} - \mu| \geq \epsilon\right) \xrightarrow{n\to\infty} 0$$

equivalently,

$$\mathbb{P}\left(|\frac{S_n}{n} - \mu| < \epsilon\right) \xrightarrow{n\to\infty} 1$$

Let

$$\xi_n = \frac{\sum_{i=1}^n (X_i - \mu)}{\sqrt{n}\sigma}, \ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{x^2}{2}} dx$$

The following theorems are of fundamental importance to option pricing.

Theorem A.3 (Central Limit Theorem (CLT))**.** *Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d random variables such that $\mathbb{E}(X_i) = \mu, \mathbb{E}(X_i^2) < \infty$ and $Var(X_i) = \sigma^2 > 0$. Then*

$$\xi_n \xrightarrow{d} \xi \sim N(0,1)$$

or,equivalently, for all $x \in (-\infty, \infty)$

$$\mathbb{P}(\xi_n \leq x) \to \Phi(x)$$

Suppose X is a random variable and we need to evaluate

$$J = \mathbb{E}(g(\mathbf{X})),$$

where $g(\mathbf{X})$ is given function. To estimate J we need to generate a sequence of independent random variables $X_1, X_2, ..., X_n$, such that $\mathbb{P}(X_i \leq x) = \mathbb{P}(X \leq x)$ and then according to $WLLN$ we have

$$J_n := \frac{1}{n}\sum_{i=1}^{n} g(X_i) \xrightarrow{\mathbb{P}} \mathbb{E}(g(X))$$

To estimate an accuracy of the approximation, we assume

$$Var(g(X)) := \sigma^2(g) < \infty$$

and note

$$Var(J_n) = \frac{Var(g(X))}{n} = \frac{\sigma^2(g)}{n}$$

Then, applying CLT we have the convergence $(J_n - J)\sqrt{n} \xrightarrow{d} N(0, \sigma^2(g))$. In particular it implies that

$$\mathbb{P}\left(|J_n - J| \leq 3\frac{\sigma(g)}{\sqrt{n}} \approx 0.997\right) \textit{ for large } n$$

The constant $\sigma^2(g)$ is usually unknown but it also can be estimated using WLLN:

$$\hat{\sigma}_n(g) := \frac{1}{n}\sum_{i=1}^{n} g^(X_i) - (J_n)^2 \xrightarrow{\mathbb{P}} \mathbb{E}(g^2(X)) - J^2 = \sigma^2(g)$$

Theorem A.4 (Strong Law of Large Number (SLLN))**.** *Let $X_1, X_2, ..., X_n$ be independent random variables with finite expected value $\mathbb{E}(X_i) = \mu$ and finite variance $V(X_i) = \sigma^2 = \sigma^2_{X_i} < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mu \textit{ a.s. and in} L^2$$

Then for any $\epsilon > 0$,

$$\mathbb{P}\left(|\frac{S_n}{n} - \mu| > \epsilon\right) \xrightarrow{n\to\infty} 0$$

In this thesis, we work under a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with filtration $\{\mathcal{F}_t, t \in [0, T]\}$

Definition A.24 (Filtration)**.** A filtration or information flow on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family of σ-algelbra $\{\mathcal{F}_t, t \in [0, T]\}$ for all $0 \leq s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.
Notation: $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$

Definition A.25 ($\mathbb{F}$-adapted)**.** A real-valued stochastic process $X = \{X(t), t \in [0, T]\}$ defined on $(\Omega, \mathbb{F}, \mathbb{P})$ is said to be $\mathbb{F}$-adapted if for any $t \in [0, T]$ the real-valued random variable $X(t)$ is $\mathcal{F}_t$−measurable, i.e. for any $x \in \mathbb{R}$ the event $\{X(t) \leq x\}$ belongs to the σ-algebra $\mathcal{F}_t$.

We usually say a process is *adapted*, rather than $\mathbb{F}$-*adapted*. In this thesis, we consider the *natural filtration*, which is the filtration generated by X.
Notation (natural filtration): $\mathcal{F}^X_t = \sigma(X_u | u < t)$ or $\mathcal{F}^X_t = \sigma\{X_s, u \leq t\}$ A stochastic process is a r.v. that is indexed by time. Time can either be discrete or continuous.

Definition A.26 (Stochastic Process)**.** A stochastic process is a parameterised collection of random variables

$$\{X(t), t \in T\}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}^d$

Definition A.27 (Stopping time)**.** A random variable $\tau \geq 0$ is said to be a stopping time or, Markov time w.r.t. $\mathcal{F}_t$ if

$$\{\tau \leq t\} \in \mathcal{F}_t \ \text{ for any } \ t \geq 0$$

Definition A.28 (Predictable process (*discrete-time*))**.** A stochastic process H is said to be predictable if H_n is H_{n-1} measurable.

Definition A.29 (Predictable process (*Continuous-time*))**.** Process H is predictable if it satisfies one of the following

1. a left-continuous adapted process, in particular, a continuous adapted process.
2. a limit (with a.s. convergence in probability) of left-continuous adapted processes.
3. a regular right-continuous process such that, for any stopping time τ, H_τ is $\mathcal{F}_{\tau-}$-measurable, the σ-field generated by sets $A \bigcap \{T < t\}$, where $A \in \mathcal{F}_t$
4. a Borel-measurable function of a predictable process

Definition A.30 (Martingale)**.** Let a process Z_t be adapted to $\mathcal{F}(t)$. We say that $Z(t)$ is a martingale with respect to $(\mathcal{F}(t), \mathbb{P})$ if

$$\mathbb{E}(Z(t)|\mathcal{F}(s)) = Z(s) \text{ for any } \ t \geq s$$

Interpretation: Martingale have no systematic stochastic drift.
Notation: $M(\mathcal{F}(t), \mathbb{P})$ is a class of all martingales w.r.t. given $\mathcal{F}(t)$ and $\mathbb{P}$

The *Stochastic Exponential* is the basic building block of the pricing framework in this thesis. This important concept is introduced as follows.
Let W be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ for a real-valued process λ, we define the real-valued $\mathbb{F}$-adapted process X by setting

$$X(t) = I(t)(\lambda) = \int_0^t \lambda(s) dW(s), \ \forall t \in [0, T]$$

The process X defined in this way is, of course, a continuous local martingale under $\mathbb{P}$ with respect to $\mathbb{F}$.

Definition A.31 (Stochastic Exponential)**.** The *stochastic exponential* (also known as the *Doléan exponential* or *exponential martingale*) of X is given by the formula, for $t \in [0, T]$,

$$\mathcal{E}(t)(X) = \mathcal{E}(t)\left(\int_0^\cdot \lambda(s) dW_s\right) = \exp\left(\int_0^t \lambda(s) dW_s - \frac{1}{2}\int_0^t |\lambda(s)|^2 ds\right),$$

that is, $\mathcal{E}(t)(X) = \exp(X(t) - \frac{1}{2}[X, X](t))$. More generally, for any continuous local martingale M we set $\mathcal{E}(t)(M) = \exp(M(t) - \frac{1}{2}[M, M](t))$ for $t \in [0, T]$.

Lemma A.5. *The stochastic exponential of X is the unique solution Y of the SDE*

$$dY(t) = Y(t)\lambda(t) dW_t = Y(t) dX(t)$$

with initial condition $Y(0) = 1$

Remark A.1. It follows immediately from Lemma A.5 that

- $d\mathcal{E}(t)(X) = \mathcal{E}(t)(X)\lambda(t)dW_t = \mathcal{E}(t)(X)dX(t)$.
- For any continuous local martingale M, the stochastic exponential $\mathcal{E}(M)$ is the unique solution Y of the SDE
$$dY(t) = Y(t)dM(t)$$
with the initial condition $Y(0) = 1$.

Definition A.32 (Markov processes)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T a fixed positive number, and let $\mathcal{F}(t)$ for $0 \le t \le T$ be a filtration of sub σ-algebra of $\mathcal{F}$. An adapted stochastic process $X(t), 0 \le t \le T$ is called Markov process w.r.t. the filtration $\mathcal{F}(t)$ for $0 \le t \le T$ if for all bounded real-valued Borel function f defined on $\mathbb{R}^d$ then

$$\mathbb{E}(f(t, X(t))|) = \mathbb{E}(f(t, X(t))|X(s)) \text{ a.s. } \forall\, 0 \le s \le t$$

Proposition A.6 (Compensated compound Poisson process)**.** *Let $X(t)$ be a compound Poisson process with jump size on $[\epsilon, 1)$ with jump measure $N(\cdot)$, the compensated compound Poisson process $X^\epsilon(t)$*

$$X^\epsilon(t) = \int_0^t \int_{\{\epsilon \le 1\}} x\tilde{N}(ds, dx) \tag{A.1}$$
$$= \int_0^t \int_{\mathbb{R}\setminus\{0\}} xN(ds, dx) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\nu(dx)ds \tag{A.2}$$
$$= \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\tilde{N}(ds, dx) \tag{A.3}$$

is a martingale.

The following definition plays an important role in probability theory.

Definition A.33 (Laplace Transform)**.** The Laplace Transform of a locally integrable function f is defined by
$$\mathcal{F}(\lambda) = \int_0^\infty e^{-\lambda x} f(x)dx$$
Laplace Transform is one of the central tool in this thesis.

Definition A.34 (moment generating function)**.** The moment generating function of a random variable is defined by
$$\varphi_X(u) = \mathbb{E}(e^{uX})$$

Notice that if X has probability density f and X is positive, then we have the Laplace Transform of (at least formally)
$$\varphi_X(-\lambda) = \mathbb{E}(e^{-\lambda X}) = \int_0^\infty e^{-\lambda x} f(x)dx$$
This is the Laplace Transform of f. It is also known as the moment generating function.

Theorem A.7 (First Fundamental Theorem of Asset Pricing)**.** *The market model defined by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the asset price $\{S(t), t \in [0, T]\}$ is arbitrage-free iff there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted assets $\{\hat{S}(t), t \in [0, T]\}$ are martingales w.r.t. $\mathbb{Q}$*

Theorem A.8 (Second Fundamental Theorem of Asset Pricing)**.** *A market defined by asst $(S_t^1, ..., S_t^m)_{t\in[0,T]}$, described as stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is complete iff there is a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.*

Theorem A.9 (No Free Lunch)**.** *We say that a price process (càdlàg process) $S = \{S_t, 0 \le t \le T\}$ admits no free lunch iff there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S is a martingale under $\mathbb{Q}$.*

A.2 Calculation of Covariance functions

The details on the calculation of the different covariance functions required in this thesis is the following. To compute the double integral of equation (5.17) we need

$$\begin{aligned}\sigma_{11} &= \operatorname{Cov}\left(a\int_t^T Y_s ds \int_t^T Y_s ds\right) = \mathbb{E}\left(a^2 \int_t^T \int_t^T Y_s Y_u du ds\right)\\ &= a^2 \int_t^T \left(\int_t^s \mathbb{E}(Y_s Y_u) du + \int_s^T \mathbb{E}(Y_s Y_u) du\right) ds\\ &= a^2 \int_t^T \left(\int_t^s \frac{1}{2\kappa} e^{-\kappa(u+s)}(e^{2\kappa u} - 1) du + \int_s^T \frac{1}{2\kappa} e^{-\kappa(u+s)}(e^{2\kappa s} - 1)\right) ds\end{aligned}$$

where the last expression can be easily computed in a symbolic package such as Mathematica.
Next

$$\sigma_{22} = \operatorname{Cov}(Y_T, Y_T) = \frac{1}{2\kappa}(1 - e^{-2\kappa T})$$

$$\sigma_{33} = \operatorname{Cov}(Y_t, Y_t) = \frac{1}{2\kappa}(1 - e^{-2\kappa t})$$

$$\sigma_{12} = \operatorname{Cov}\left(a\int_t^T Y_s ds, Y_T\right) = \int_t^T \frac{a}{2\kappa}(e^{2\kappa t} - 1) ds$$

$$\sigma_{23} = \operatorname{Cov}(Y_T, T_t) = \frac{1}{2\kappa} e^{-\kappa(t+T)}(e^{2\kappa t} - 1)$$

Appendix B

Mathematica codes of the analytical approximation and MATLAB codes of simulations for the main (Geometric Lévy) model

The following implements the semi-analytical method presented previously to obtain the price of VWAP options. We utilise the Mathematica codes first developed by Tim Ling.

To keep things simple, we compute moments given one set of parameters as following; start with *notebook GL 2nd moment basic_rho0.nb*

```
S0=110; σ=5; T=1; λ=2; b=0; ρ=0; σVG=0.1; σBM=0.1; σRHO=σBM*√(1 − ρ^2);
```

Following chapter 5.4 in thesis, we have the following:

$$m[\mathrm{t\_}] = ae^{-\lambda t} + a\left(1 - e^{-\lambda t}\right);$$

Needs["MultivariateStatistics`"]

$$\mathrm{PDF}\left[\mathrm{MultinormalDistribution}\left[\left\{\mu\mathrm{x1} + \frac{\sigma 13}{\sigma 33}(z - \mu\mathrm{x3}), \mu\mathrm{x2} + \frac{\sigma 23}{\sigma 33}(z - \mu\mathrm{x3})\right\},\right.\right.$$
$$\left.\left\{\left\{\sigma 11 - \frac{(\sigma 13)^2}{\sigma 33}, \sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}\right\}, \left\{\sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}, \sigma 22 - \frac{(\sigma 23)^2}{\sigma 33}\right\}\right\}\right], \{x, y\}\Big];$$

$$\mathrm{expon}[\mathrm{x\_}, \mathrm{y\_}] = \frac{1}{2}\left(-\left(\frac{\left(y - \mu\mathrm{x2} - \frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\left(-\sigma 12 + \frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.\right.$$
$$\left. + \frac{\left(x - \mu\mathrm{x1} - \frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(\sigma 22 - \frac{\sigma 23^2}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(x - \mu\mathrm{x1} - \frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)$$
$$-\left(\frac{\left(\sigma 11 - \frac{\sigma 13^2}{\sigma 33}\right)\left(y - \mu\mathrm{x2} - \frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.$$
$$\left.\left. + \frac{\left(x - \mu\mathrm{x1} - \frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(-\sigma 12 + \frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(y - \mu\mathrm{x2} - \frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\right)$$
$$-2q\sigma x + \frac{(\kappa-\lambda)y^2}{2}; \text{(*!!!!!!!!!!!!!!!!!!!!!!!!!!*)}$$

$$G = \mathrm{FullSimplify}[\mathrm{expon}[0, 0]];$$

$$F = \mathrm{FullSimplify}[\mathrm{Coefficient}[\mathrm{expon}[x, y], xy]];$$

$$A = -\mathrm{FullSimplify}\left[\mathrm{Coefficient}\left[\mathrm{expon}[x, 0], x^2\right]\right];$$

B = FullSimplify[Coefficient[expon[0, y], y^2]];

Cx = FullSimplify[Coefficient[expon[x, 0], x]];

Dy = FullSimplify[Coefficient[expon[0, y], y]];

JYt = FullSimplify$\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G, z\right]\right]$;

HYt2 = FullSimplify$\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G, z^2\right]\right]$;

constZ[z_] = $\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G$;

L = FullSimplify[constZ[0]];

Check that the coefficient H, J, L are correct

FullSimplify$\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G-\text{HYt2}*z^2-\text{JYt}*z-L\right]$

0

Check that the coefficient A, B, C, D, F are correct

FullSimplify$\left[\text{expon}[x,y]-\left((\text{Cx}*x)+(\text{Dy}*y)+(F*(xy))+\left(-A*x^2\right)+\left(B*y^2\right)+G\right)\right]$

0

OC1 = FullSimplify$\left[\frac{2\pi}{\sqrt{A}\sqrt{-\frac{4AB+F^2}{A}}}\bigg/\left(2\pi\sqrt{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\right]$;

Now the outer integral

*)

PDF[MultinormalDistribution[{μa, μb}, {{σaa, σab}, {σab, σbb}}], {x, y}];

expon1[x_, y_] = $\frac{1}{2}\left(-(y-\mu\text{b})\left(\frac{(y-\mu\text{b})\sigma\text{aa}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}-\frac{(x-\mu\text{a})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)-(x-\mu\text{a})\left(-\frac{(y-\mu\text{b})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}+\frac{(x-\mu\text{a})\sigma\text{bb}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right)$

+(c1)x + (c2)x^2 + (c3)y;

G1 = FullSimplify[expon1[0, 0]];

F1 = FullSimplify[Coefficient[expon1[x, y], xy]];

A1 = −FullSimplify[Coefficient[expon1[x, 0], x^2]];

B1 = FullSimplify[Coefficient[expon1[0, y], y^2]];

Cx1 = FullSimplify[Coefficient[expon1[x, 0], x]];

Dy1 = FullSimplify[Coefficient[expon1[0, y], y]];

(*CheckthatthecoefficientsA1, B1, C1, D1, F1andG1arecorrect*)

FullSimplify$\left[\text{expon1}[x,y]-\left((\text{Cx1}*x)+(\text{Dy1}*y)+(\text{F1}*(xy))+\left(-\text{A1}*x^2\right)+\left(\text{B1}*y^2\right)+\text{G1}\right)\right]$

0

OC2 = FullSimplify$\left[\frac{2\pi}{\sqrt{\text{A1}}\sqrt{-\frac{4\text{A1B1}+\text{F1}^2}{\text{A1}}}}\bigg/\left(2\pi\sqrt{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right]$;

(*Now need to collect coefficients for the triple integral*)

exponTrip[u_, v_, w_] = $\frac{1}{2}\left(-w\left(\frac{w(-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.\right.$

$$+\frac{v(\sigma\mathrm{ab}\sigma\mathrm{ac}-\sigma\mathrm{aa}\sigma\mathrm{bc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$

$$+\frac{u(-\sigma\mathrm{ac}\sigma\mathrm{bb}+\sigma\mathrm{ab}\sigma\mathrm{bc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\Bigg)$$

$$-v\Bigg(\frac{w(\sigma\mathrm{ab}\sigma\mathrm{ac}-\sigma\mathrm{aa}\sigma\mathrm{bc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$

$$+\frac{v(-\sigma\mathrm{ac}^2+\sigma\mathrm{aa}\sigma\mathrm{cc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$

$$+\frac{u(\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{ab}\sigma\mathrm{cc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\Bigg)$$

$$-u\Bigg(\frac{w(-\sigma\mathrm{ac}\sigma\mathrm{bb}+\sigma\mathrm{ab}\sigma\mathrm{bc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$

$$+\frac{v(\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{ab}\sigma\mathrm{cc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$

$$+\frac{u(-\sigma\mathrm{bc}^2+\sigma\mathrm{bb}\sigma\mathrm{cc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\Bigg)\Bigg)$$

$$+(-2\sigma m[t]z+J)u+\left(-z\sigma^2+H\right)u^2-2q\sigma v+(-2\sigma\ m[s]r)w+\left(-r\sigma^2\right)w^2;$$

C00 = FullSimplify[exponTrip[0, 0, 0]];

Cu2 = −Coefficient [exponTrip[u, 0, 0], u^2] ;

Cv2 = Coefficient [exponTrip[0, v, 0], v^2] ;

Cw2 = Coefficient [exponTrip[0, 0, w], w^2] ;

Cu = Coefficient[exponTrip[u, 0, 0], u];

Cv = Coefficient[exponTrip[0, v, 0], v];

Cw = Coefficient[exponTrip[0, 0, w], w];

Cuv = Coefficient[exponTrip[u, v, 0], uv];

Cuw = Coefficient[exponTrip[u, 0, w], uw];

Cvw = Coefficient[exponTrip[0, v, w], vw];

FullSimplify [exponTrip[u, v, w] − (−Cu2 ∗ u^2 + Cv2 ∗ v^2 + Cw2 ∗ w^2 + Cu ∗ u + Cv ∗ v

+Cw ∗ w + Cuv ∗ (u ∗ v) + Cuw ∗ (u ∗ w) + Cvw ∗ (v ∗ w) + C00)]

0

$\kappa = \sqrt{\lambda^2 + 2q\sigma^2}$;

J = JYt;

H = HYt2;

c1 = $-2zm[t]\sigma + J$; (*!!!!!!!!!!!!!!!!!!!!!*)

c2 = $-z\sigma^2 + H$;

c3 = $-2q\sigma$;

gamma[z_, q_] = OC1 ∗ OC2 ∗ Exp[L] ∗ Exp $\left[\frac{\mathrm{B1(Cx1)^2-Dy1(A1Dy1+Cx1\ F1)}}{\mathrm{4*(A1B1)+F1^2}}+\mathrm{G1}\right]$;

(*Now to compute all the covariance*)

$\sigma 22 = \frac{1}{2\kappa}\left(1-e^{-2\kappa T}\right)$;

$\sigma 33 = \frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right)$;

$$\sigma 12 = \int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+T)}\left(e^{2\kappa s}-1\right)\right)ds;$$
$$\sigma 13 = \int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa t}-1\right)\right)ds;$$
$$\sigma 23 = \frac{1}{2\kappa}\left(e^{\kappa(t-T)}-e^{-\kappa(t+T)}\right);$$
$$\sigma\text{aa} = \frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right);$$
$$\sigma\text{ab} = \int_0^t m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s}-1\right)\right)ds;$$

(*σ11 and σbb were precomputed*)

$$\sigma 11 = -\frac{a^2\left(2+e^{-2t\kappa}-2e^{(t-T)\kappa}+e^{-2T\kappa}-2e^{-(t+T)\kappa}+2t\kappa-2T\kappa\right)}{2\kappa^3};$$
$$\sigma\text{bb} = \frac{a^2e^{-2t\kappa}\left(-1+4e^{t\kappa}+e^{2t\kappa}(-3+2t\kappa)\right)}{2\kappa^3};$$
$$\sigma\text{cc} = \frac{1}{2\kappa}\left(1-e^{-2\kappa s}\right);$$
$$\sigma\text{ac} = \frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s}-1\right);\text{(* s} \le \text{t*)}$$
$$\sigma\text{bc} = -\frac{ae^{-(s+t)\kappa}\left(-1+e^{s\kappa}\right)\left(1+e^{s\kappa}-2e^{t\kappa}\right)}{2\kappa^2};$$

Clear[S0, n, g, σVG, σBM, θ, v, μ];

$$\psi[\text{n\_}]:=\left(n*g+\frac{1}{2}n\text{\^{}}2\sigma\text{RHO\^{}}2-\text{Log}\left[1-n\theta v+\frac{(n\sigma\text{VG})\text{\^{}}2v}{2}\right]\Big/v\right)$$

cf[n_]:=Exp[$t*\psi[n]$]

(*Compute first moment*) S0 = 110;

$\sigma = 5;$

$T = 1;$

$\lambda = 2;$

$b = 0;$

$a = 22;$

$\rho = 0;$

σVG = 0.1; (*control parameter for skewness*)

σBM = 0.1; (*this is the diffusion coefficient in the vg stock price model*)

$\sigma\text{RHO} = \sigma\text{BM} * \left(\sqrt{1-\rho\text{\^{}}2}\right);$

$v = 0.1;$ (*control parameter for kurtosis*)

$\theta = -0.14;$ (*measure of symmetry*)

$\mu = 0.1;$

$$g = \mu - 0.5*\sigma\text{BM\^{}}2 + \text{Log}\left[1-\theta v+\frac{(\sigma\text{VG})\text{\^{}}2v}{2}\right]\Big/v;$$

μx1 = 0;

μx2 = 0;

μx3 = 0;

μa = 0;

μb = 0;

$(*\psi[1];*)$

$\text{Phi}[\text{z\_}, \text{q\_}] = \text{Exp}\left[-z(m[t])^2 - q\int_0^T (m[s])^2\, ds + \frac{(\lambda-\kappa)T}{2} - z*b - q*b*T\right] * \text{gamma}[z,q];$

$\text{dzPhi}[\text{q\_}, \text{t\_}] = \text{Derivative}[1,0][\text{Phi}][z,q]/.z \to 0;$

$(*\text{moment1} = \text{NIntegrate}\left[-\text{S0} * e^{(\mu+0.5*\rho\text{\^{}}2*\sigma\text{BM\^{}}2)*t}\text{dzPhi}[q,t], \{q,0,\infty\}, \{t,0,T\}\right];*)$

$\text{moment1} = \text{NIntegrate}[-\text{S0} * \text{cf}[1]\text{Exp}[0.5 * \rho\text{\^{}}2 * \sigma\text{BM\^{}}2 * t]\text{dzPhi}[q,t], \{q,0,\infty\}, \{t,0,T\}]$

TimeUsed[]

115.681

11.651

Now to set up the second moment

$\text{OC2trip} = \frac{1}{\sqrt{\text{Cu2}}\sqrt{-\frac{\text{Cuv}^2+4\text{Cu2Cv2}}{\text{Cu2}}}}2\pi/$
$\left(\sqrt{\frac{-\text{Cuw}^2\text{Cv2}+\text{CuvCuwCvw}+\text{Cu2Cvw}^2-\text{Cuv}^2\text{Cw2}-4\text{Cu2Cv2Cw2}}{\text{Cuv}^2\pi+4\text{Cu2Cv2}\pi}}\right)/$
$\left(2\sqrt{2}\pi^{3/2}\sqrt{-\sigma\text{ac}^2\sigma\text{bb} + 2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc} - \sigma\text{aa}\sigma\text{bc}^2 - \sigma\text{ab}^2\sigma\text{cc} + \sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right);$

$\text{gamma2}[\text{z\_}, \text{r\_}, \text{q\_}] = \text{OC1} * \text{OC2trip} * \text{Exp}[L] * \text{Exp}\left[-\left(\text{Cuw}^2\left(\text{Cv}^2 - 4\text{C00Cv2}\right) + 4\text{C00Cu2Cvw}^2\right.\right.$

$-4\text{Cu2CvCvwCw} + \text{Cuv}^2\text{Cw}^2$

$+4\text{Cu2Cv2Cw}^2 + \text{Cuw}(4\text{C00CuvCvw} - 2\text{CuCvCvw} - 2\text{CuvCvCw} + 4\text{CuCv2Cw})$

$-4\text{C00Cuv}^2\text{Cw2} + 4\text{Cu2Cv}^2\text{Cw2}$

$-16\text{C00Cu2Cv2Cw2} + \text{CuCuv}(-2\text{CvwCw} + 4\text{CvCw2}) + \text{Cu}^2\left(\text{Cvw}^2 - 4\text{Cv2Cw2}\right))/$

$\left.\left(4\left(\text{Cuw}^2\text{Cv2} - \text{CuvCuwCvw} - \text{Cu2Cvw}^2 + \text{Cuv}^2\text{Cw2} + 4\text{Cu2Cv2Cw2}\right)\right)\right];$

$\text{Phi2}[\text{z\_}, \text{r\_}, \text{q\_}] = \text{Exp}\left[-z(m[t])^2 - r(m[s])^2 - q\int_0^T (m[s])^2\, ds + \frac{(\lambda-\kappa)T}{2}\right] * \text{gamma2}[z,r,q];$

$\text{dzdrPhi2}[\text{q\_}, \text{t\_}, \text{s\_}] = ((\text{Derivative}[1,1,0][\text{Phi2}][z,r,q]/.z \to 0)/.r \to 0);$

moment2 =

$\text{NIntegrate}\left[\text{S0}^2 q\text{Exp}[s * \psi[2] + (t-s) * \psi[1] + 0.5 * \rho\text{\^{}}2 * \sigma\text{BM\^{}}2 * (s+t)]\text{dzdrPhi2}[q,t,s],\right.$

$\{q,0,\infty\}, \{t,0,T\}, \{s,0,t\}] +$

$\text{NIntegrate}\left[\text{S0}^2 q\text{Exp}[t * \psi[2] + (s-t) * \psi[1] + 0.5 * \rho\text{\^{}}2 * \sigma\text{BM\^{}}2 * (s+t)]\text{dzdrPhi2}[q,s,t],\right.$

$\{q,0,\infty\}, \{t,0,T\}, \{s,t,T\}];$

TimeUsed[]

Print["First moment = " <> ToString[moment1]]

Print["Second moment = " <> ToString[moment2]]

Print["Variance = " <> ToString[moment2 − moment1^2]]

Print["SD = " <> ToString[Sqrt[moment2 − moment1^2]]]

Print["mu tilde = " <> ToString[Log[moment1/S0]/T]]

Print$\left[\text{“sig tilde = ”} <> \text{ToString}\left[\text{Sqrt}\left[\text{Log}\left[\frac{(\text{moment2}-\text{moment1\^{}2})+(\text{moment1})^2}{(\text{moment1})^2}\right]\Big/T\right]\right]\right]$

First moment = 115.681

Second moment = 13392.7

Variance = 10.7393

SD = 3.27709

mu tilde = 0.0503521

sig tilde = 0.0283231

To see how $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$ evolve over time, we use *notebook GL LNParEvoln plot_rho0.nb*. Everything has been done previously follows.

```
S0=110; σ=5; T=1; λ=2; b=0; ρ=0; σVG=0.1; σBM=0.1; σRHO=σBM*√(1 - ρ^2);
```

Following chapter 5.4 in thesis, we have the following:

$m[\text{t\_}] = ae^{-\lambda t} + a\left(1 - e^{-\lambda t}\right);$

Needs[“MultivariateStatistics”]

$\text{PDF}\left[\text{MultinormalDistribution}\left[\left\{\mu\text{x1} + \frac{\sigma 13}{\sigma 33}(z - \mu\text{x3}), \mu\text{x2} + \frac{\sigma 23}{\sigma 33}(z - \mu\text{x3})\right\},\right.\right.$
$\left.\left.\left\{\left\{\sigma 11 - \frac{(\sigma 13)^2}{\sigma 33}, \sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}\right\}, \left\{\sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}, \sigma 22 - \frac{(\sigma 23)^2}{\sigma 33}\right\}\right\}\right], \{x, y\}\right];$

$$\text{expon}[\text{x\_}, \text{y\_}] = \frac{1}{2}\left(-\left(\frac{\left(y-\mu\text{x2}-\frac{(z-\mu\text{x3})\sigma 23}{\sigma 33}\right)\left(-\sigma 12+\frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.\right.$$
$$\left.+\frac{\left(x-\mu\text{x1}-\frac{(z-\mu\text{x3})\sigma 13}{\sigma 33}\right)\left(\sigma 22-\frac{\sigma 23^2}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(x - \mu\text{x1} - \frac{(z-\mu\text{x3})\sigma 13}{\sigma 33}\right)$$
$$-\left(\frac{\left(\sigma 11-\frac{\sigma 13^2}{\sigma 33}\right)\left(y-\mu\text{x2}-\frac{(z-\mu\text{x3})\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.$$
$$\left.\left.+\frac{\left(x-\mu\text{x1}-\frac{(z-\mu\text{x3})\sigma 13}{\sigma 33}\right)\left(-\sigma 12+\frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(y - \mu\text{x2} - \frac{(z-\mu\text{x3})\sigma 23}{\sigma 33}\right)\right)$$
$$-2q\sigma x + \frac{(\kappa-\lambda)y^2}{2}; \text{(*!!!!!!!!!!!!!!!!!!!!!!!!!!*)}$$

$G = \text{FullSimplify}[\text{expon}[0,0]];$

$F = \text{FullSimplify}[\text{Coefficient}[\text{expon}[x,y], xy]];$

$A = -\text{FullSimplify}\left[\text{Coefficient}\left[\text{expon}[x,0], x^2\right]\right];$

$B = \text{FullSimplify}\left[\text{Coefficient}\left[\text{expon}[0,y], y^2\right]\right];$

$\text{Cx} = \text{FullSimplify}[\text{Coefficient}[\text{expon}[x,0], x]];$

$\text{Dy} = \text{FullSimplify}[\text{Coefficient}[\text{expon}[0,y], y]];$

$\text{JYt} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2} + G, z\right]\right];$

$\text{HYt2} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2} + G, z^2\right]\right];$

$\text{constZ}[\text{z\_}] = \frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2} + G;$

$L = \text{FullSimplify}[\text{constZ}[0]];$

Check that the coefficient H, J, L are correct

$\text{FullSimplify}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2} + G - \text{HYt2} * z^2 - \text{JYt} * z - L\right]$

0

Check that the coefficient A, B, C, D, F are correct

$$\text{FullSimplify}\left[\text{expon}[x,y] - \left((\text{Cx} * x) + (\text{Dy} * y) + (F * (xy)) + \left(-A * x^2\right) + \left(B * y^2\right) + G\right)\right]$$

0

$$\text{OC1} = \text{FullSimplify}\left[\frac{2\pi}{\sqrt{A}\sqrt{-\frac{4AB+F^2}{A}}} / \left(2\pi\sqrt{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\right];$$

Now the outer integral

*)

$$\text{PDF}[\text{MultinormalDistribution}[\{\mu\text{a}, \mu\text{b}\}, \{\{\sigma\text{aa}, \sigma\text{ab}\}, \{\sigma\text{ab}, \sigma\text{bb}\}\}], \{x, y\}];$$

$$\text{expon1}[\text{x\_}, \text{y\_}] = \frac{1}{2}\left(-(y-\mu\text{b})\left(\frac{(y-\mu\text{b})\sigma\text{aa}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}} - \frac{(x-\mu\text{a})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right) - (x-\mu\text{a})\left(-\frac{(y-\mu\text{b})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}} + \frac{(x-\mu\text{a})\sigma\text{bb}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right)$$
$$+(\text{c1})x + (\text{c2})x^2 + (\text{c3})y;$$

$$\text{G1} = \text{FullSimplify}[\text{expon1}[0,0]];$$

$$\text{F1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,y], xy]];$$

$$\text{A1} = -\text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[x,0], x^2\right]\right];$$

$$\text{B1} = \text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[0,y], y^2\right]\right];$$

$$\text{Cx1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,0], x]];$$

$$\text{Dy1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[0,y], y]];$$

(*CheckthatthecoefficientsA1, B1, C1, D1, F1andG1arecorrect*)

$$\text{FullSimplify}\left[\text{expon1}[x,y] - \left((\text{Cx1} * x) + (\text{Dy1} * y) + (\text{F1} * (xy)) + \left(-\text{A1} * x^2\right) + \left(\text{B1} * y^2\right) + \text{G1}\right)\right]$$

0

$$\text{OC2} = \text{FullSimplify}\left[\frac{2\pi}{\sqrt{\text{A1}}\sqrt{-\frac{4\text{A1B1}+\text{F1}^2}{\text{A1}}}} / \left(2\pi\sqrt{-\sigma\text{ab}^2 + \sigma\text{aa}\sigma\text{bb}}\right)\right];$$

(*Now need to collect coefficients for the triple integral*)

$$\text{exponTrip}[\text{u\_}, \text{v\_}, \text{w\_}] = \frac{1}{2}\left(-w\left(\frac{w(-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.\right.$$
$$+\frac{v(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$$
$$\left.+\frac{u(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$$
$$-v\left(\frac{w(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.$$
$$+\frac{v(-\sigma\text{ac}^2+\sigma\text{aa}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$$
$$\left.+\frac{u(\sigma\text{ac}\sigma\text{bc}-\sigma\text{ab}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$$
$$-u\left(\frac{w(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.$$
$$+\frac{v(\sigma\text{ac}\sigma\text{bc}-\sigma\text{ab}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$$
$$\left.\left.+\frac{u(-\sigma\text{bc}^2+\sigma\text{bb}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)\right)$$
$$+(-2\sigma m[t]z + J)u + \left(-z\sigma^2 + H\right)u^2 - 2q\sigma v + (-2\sigma\ m[s]r)w + \left(-r\sigma^2\right)w^2;$$

$$\text{C00} = \text{FullSimplify}[\text{exponTrip}[0,0,0]];$$

$$\text{Cu2} = -\text{Coefficient}\left[\text{exponTrip}[u,0,0], u^2\right];$$

$\text{Cv2} = \text{Coefficient}\left[\text{exponTrip}[0, v, 0], v^2\right];$

$\text{Cw2} = \text{Coefficient}\left[\text{exponTrip}[0, 0, w], w^2\right];$

$\text{Cu} = \text{Coefficient}[\text{exponTrip}[u, 0, 0], u];$

$\text{Cv} = \text{Coefficient}[\text{exponTrip}[0, v, 0], v];$

$\text{Cw} = \text{Coefficient}[\text{exponTrip}[0, 0, w], w];$

$\text{Cuv} = \text{Coefficient}[\text{exponTrip}[u, v, 0], uv];$

$\text{Cuw} = \text{Coefficient}[\text{exponTrip}[u, 0, w], uw];$

$\text{Cvw} = \text{Coefficient}[\text{exponTrip}[0, v, w], vw];$

$\text{FullSimplify}\left[\text{exponTrip}[u, v, w] - \left(-\text{Cu2} * u^2 + \text{Cv2} * v^2 + \text{Cw2} * w^2 + \text{Cu} * u + \text{Cv} * v\right.\right.$

$\left.\left.+\text{Cw} * w + \text{Cuv} * (u * v) + \text{Cuw} * (u * w) + \text{Cvw} * (v * w) + \text{C00}\right)\right]$

0

$\kappa = \sqrt{\lambda^2 + 2q\sigma^2};$

$J = \text{JYt};$

$H = \text{HYt2};$

$\text{c1} = -2zm[t]\sigma + J;$ (*!!!!!!!!!!!!!!!!!!!!!*)

$\text{c2} = -z\sigma^2 + H;$

$\text{c3} = -2q\sigma;$

$\text{gamma}[\text{z\_}, \text{q\_}] = \text{OC1} * \text{OC2} * \text{Exp}[L] * \text{Exp}\left[\frac{\text{B1}(\text{Cx1})^2 - \text{Dy1}(\text{A1Dy1} + \text{Cx1 F1})}{4 * (\text{A1B1}) + \text{F1}^2} + \text{G1}\right];$

(*Now to compute all the covariance*)

$\sigma 22 = \frac{1}{2\kappa}\left(1 - e^{-2\kappa T}\right);$

$\sigma 33 = \frac{1}{2\kappa}\left(1 - e^{-2\kappa t}\right);$

$\sigma 12 = \int_t^T m[s]\left(\frac{1}{2\kappa} e^{-\kappa(s+T)}\left(e^{2\kappa s} - 1\right)\right) ds;$

$\sigma 13 = \int_t^T m[s]\left(\frac{1}{2\kappa} e^{-\kappa(s+t)}\left(e^{2\kappa t} - 1\right)\right) ds;$

$\sigma 23 = \frac{1}{2\kappa}\left(e^{\kappa(t-T)} - e^{-\kappa(t+T)}\right);$

$\sigma \text{aa} = \frac{1}{2\kappa}\left(1 - e^{-2\kappa t}\right);$

$\sigma \text{ab} = \int_0^t m[s]\left(\frac{1}{2\kappa} e^{-\kappa(s+t)}\left(e^{2\kappa s} - 1\right)\right) ds;$

(*σ11 and σbb were precomputed*)

$\sigma 11 = -\frac{a^2\left(2 + e^{-2t\kappa} - 2e^{(t-T)\kappa} + e^{-2T\kappa} - 2e^{-(t+T)\kappa} + 2t\kappa - 2T\kappa\right)}{2\kappa^3};$

$\sigma \text{bb} = \frac{a^2 e^{-2t\kappa}\left(-1 + 4e^{t\kappa} + e^{2t\kappa}(-3 + 2t\kappa)\right)}{2\kappa^3};$

$\sigma \text{cc} = \frac{1}{2\kappa}\left(1 - e^{-2\kappa s}\right);$

$\sigma \text{ac} = \frac{1}{2\kappa} e^{-\kappa(s+t)}\left(e^{2\kappa s} - 1\right);$ (* s ≤ t*)

$\sigma\text{bc} = -\frac{a e^{-(s+t)\kappa}(-1+e^{s\kappa})(1+e^{s\kappa}-2e^{t\kappa})}{2\kappa^2};$

$\text{Clear}[\text{S0}, n, g, \sigma\text{VG}, \sigma\text{BM}, \theta, v, \mu];$

$\psi[\text{n\_}]{:=}\left(n * g + \frac{1}{2}n\text{^}2\sigma\text{RHO^}2 - \text{Log}\left[1 - n\theta v + \frac{(n\sigma\text{VG})\text{^}2v}{2}\right]\Big/ v\right)$

$\text{cf}[\text{n\_}]{:=}\text{Exp}[t * \psi[n]]$

(*Compute first moment*) $\text{S0} = 110;$

$\sigma = 5;$

$T = 1;$

$\lambda = 2;$

$b = 0;$

$a = 22;$

$\rho = 0;$

σVG = 0.1; (*control parameter for skewness*)

σBM = 0.1; (*this is the diffusion coefficient in the vg stock price model*)

$\sigma\text{RHO} = \sigma\text{BM} * \left(\sqrt{1 - \rho\text{^}2}\right);$

v = 0.1; (*control parameter for kurtosis*)

θ = −0.14; (*measure of symmetry*)

$\mu = 0.1;$

$g = \mu - 0.5 * \sigma\text{BM^}2 + \text{Log}\left[1 - \theta v + \frac{(\sigma\text{VG})\text{^}2v}{2}\right]\Big/ v;$

$\mu\text{x1} = 0;$

$\mu\text{x2} = 0;$

$\mu\text{x3} = 0;$

$\mu\text{a} = 0;$

$\mu\text{b} = 0;$

Now, the moments derived in Chapter 5 are solved for each time interval $[0, 1]$. At each time, the approximations to the mean (equation 5.5) and variance (5.6) of the VWAP are used to obtain the values of $\tilde{\mu}(t)$ and $\tilde{\sigma}(t)$. In Mathematica, we have the following

Clear[muPoints, sigmaPoints, xPoints, muTilde, sigmaTilde]

TEnd = 1;

incrementSize = 0.01;

$i = 1;$

Do[

$\text{Phi}[\text{z\_}, \text{q\_}] = \text{Exp}\left[-z(m[t])^2 - q\int_0^T (m[s])^2\, ds + \frac{(\lambda-\kappa)T}{2} - z * b - q * b * T\right] * \text{gamma}[z, q];$

$\text{dzPhi}[\text{q\_}, \text{t\_}] = \text{Derivative}[1, 0][\text{Phi}][z, q]/.z \to 0;$

```
moment1 = NIntegrate[-S0 * cf[1]Exp[0.5 * rho^2 * sigmaBM^2 * t]dzPhi[q, t], {q, 0, ∞}, {t, 0, T}];
(*
Nowtosetupthesecondmoment
*)

OC2trip = 1/(Sqrt[Cu2] Sqrt[-(Cuv^2+4Cu2Cv2)/Cu2]) 2π/
(Sqrt[(-Cuw^2Cv2+CuvCuwCvw+Cu2Cvw^2-Cuv^2Cw2-4Cu2Cv2Cw2)/(Cuv^2π+4Cu2Cv2π)])/
(2Sqrt[2]π^(3/2) Sqrt[-σac^2σbb + 2σabσacσbc - σaaσbc^2 - σab^2σcc + σaaσbbσcc]);
gamma2[z_, r_, q_] = OC1 * OC2trip * Exp[L] * Exp[-(Cuw^2 (Cv^2 - 4C00Cv2) + 4C00Cu2Cvw^2
-4Cu2CvCvwCw + Cuv^2Cw^2
+4Cu2Cv2Cw^2 + Cuw(4C00CuvCvw - 2CuCvCvw - 2CuvCvCw + 4CuCv2Cw)
-4C00Cuv^2Cw2 + 4Cu2Cv^2Cw2
 -16C00Cu2Cv2Cw2 + CuCuv(-2CvwCw + 4CvCw2) + Cu^2 (Cvw^2 - 4Cv2Cw2))/
(4 (Cuw^2Cv2 - CuvCuwCvw - Cu2Cvw^2 + Cuv^2Cw2 + 4Cu2Cv2Cw2))];
Phi2[z_, r_, q_] = Exp[-z(m[t])^2 - r(m[s])^2 - q ∫_0^T (m[s])^2 ds + ((λ-κ)T)/2] * gamma2[z, r, q];
dzdrPhi2[q_, t_, s_] = ((Derivative[1, 1, 0][Phi2][z, r, q]/.z → 0)/.r → 0);
moment2 = NIntegrate[S0^2 q Exp[s * ψ[2] + (t - s) * ψ[1] + 0.5 * rho^2 * sigmaBM^2 * (s + t)]dzdrPhi2[q, t, s],
{q, 0, ∞}, {t, 0, T}, {s, 0, t}]+
NIntegrate[S0^2 q Exp[t * ψ[2] + (s - t) * ψ[1] + 0.5 * rho^2 * sigmaBM^2 * (s + t)]dzdrPhi2[q, s, t],
{q, 0, ∞}, {t, 0, T}, {s, t, T}];
muTilde[i] = Log[moment1/S0]/T;
sigmaTilde[i] = Sqrt[Log[((moment2-moment1^2)+(moment1)^2)/(moment1)^2]/T];
i++,
{T, incrementSize, TEnd, incrementSize}]
(*ENDDO - LOOP*)

(* Plot graphs*)
xPoints = Table[T, {T, incrementSize, TEnd, incrementSize}];
muPoints = Table[{xPoints[[i]], muTilde[i]}, {i, 1, Length[xPoints]}];
sigmaPoints = Table[{xPoints[[i]], sigmaTilde[i]}, {i, 1, Length[xPoints]}];
ListLinePlot[muPoints, PlotRange → Full, AxesLabel → {Style[T, Large], Style["mu", Large]}]
ListLinePlot[sigmaPoints, PlotRange → Full, AxesLabel → {Style[T, Large], Style["sigma", Large]}]

muTilde[100] - muTilde[1]
sigmaTilde[100] - sigmaTilde[1]

0.000351263
```

0.000486757

?muTilde

?sigmaTilde

Global`muTilde

Print["First moment = " <> ToString[moment1]]

Print["Second moment = " <> ToString[moment2]]

Print["Variance = " <> ToString[moment2 – moment1^2]]

Print["SD = " <> ToString[Sqrt[moment2 – moment1^2]]]

Print["mu tilde = " <> ToString[Log[moment1/S0]/1]]

Print["sig tilde = " <> ToString[Sqrt[Log[$\frac{(\text{moment2}-\text{moment1\^{}2})+(\text{moment1})^2}{(\text{moment1})^2}$]/1]]]

First moment = 115.681
Second moment = 13392.7
Variance = 10.7393
SD = 3.27709
mu tilde = 0.0503521
sig tilde = 0.0283231

To evaluate the accuracy of our methods in pricing the VWAP option, we need more prices. If 100 prices are needed, then 200 (300) moments values are requires in lognormal (GIG) case. The *GL 2nd moment 100_rho0.nb* notebook was coded in exactly the same manner as *GL 2nd moment basic_rho0.nb* except that the diffusion parameter σBM now become vectored-valued: σBM = 1 is partitioned into 100 interval. The moments were computed similarly as the last two notebook and the data was exported for the next job.

The *LNprice_rho0.nb* notebook computes option price via the lognormal approximation, *first*, import the moments data.

```
csvFile=Import["F:\\Thesis codes\\FinalMoment\\GL 2nd moment 100_rho0.csv",
Transpose[{sig,m1,m2}],"CSV"];

sigmaPrice=Table[csvFile[[i]][[1]],{i,2,Length[csvFile]}];
mu1=Table[csvFile[[i]][[2]],{i,2,Length[csvFile]}];
mu2=Table[csvFile[[i]][[3]],{i,2,Length[csvFile]}];
```

here mu1 and mu2 are the first two moments of VWAP, using equations 5.5 and 5.6, $\tilde{\mu}$ and $\tilde{\sigma}$ are solved for each σBM from 0.01 to 1, i.e.

```
Clear[muTilde,sigmaTilde]

muTilde=Log[mu1/S0]/1
sigmaTilde=Sqrt[Log[((mu2-mu1^2)+(mu1)^2)/(mu1)^2]/1]

{0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
```

```
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,0.0503521,
0.0503521,0.0503521,0.0503521,0.0503521}
```

```
{0. +0.0507952 I,0. +0.0497769 I,0. +0.0480316 I,0. +0.0454742 I,0. +0.0419607 I,
0. +0.0372157 I,0. +0.0306744 I,0. +0.0206977I,0.0123571,0.0283231,
0.0389921,0.0480339,0.0562396,0.0639327,0.0712799,0.0783792,0.0852932,0.0920643,0.0987224,
0.10529,0.111783,0.118214,0.124594,0.130931,0.137231,0.143501,0.149743,0.155962,0.162162,
0.168345,0.174513,0.180669,0.186814,0.19295,0.199079,0.205201,0.211318,0.217431,0.22354,
0.229648,0.235753,0.241858,0.247962,0.254067,0.260173,0.26628,0.272389,0.278501,0.284615,
0.290733,0.296854,0.30298,0.30911,0.315244,0.321384,0.327529,0.33368,0.339837,0.346,0.35217,
0.358346,0.36453,0.370721,0.376919,0.383126,0.38934,0.395563,0.401794,0.408034,0.414283,
0.420541,0.426809,0.433086,0.439373,0.44567,0.451977,0.458295,0.464623,0.470962,0.477313,
0.483674,0.490047,0.496431,0.502827,0.509235,0.515655,0.522088,0.528533,0.53499,0.541461,
0.547944,0.554441,0.560951,0.567475,0.574012,0.580563,0.587129,0.593708,0.600302,0.606911}
```

Note that the first eight roots of $\tilde{\sigma}$ are complex and so were taken out for the later integration. Now using the lognormal distribution as the state price density, the prices could be computed by numerical integration via NIntegrate.

$$f[\mathrm{x\_}, \mu\_, \sigma\_] := \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{Exp}\left[-\left(x - \mathrm{Log}[110] - \left(\mu - 0.5 * \sigma^2\right)\right)\hat{}2/\left(2 * \sigma^2\right)\right]$$

```
KStart=100;
KEnd=120;
LNPrices=Flatten[Table[Table[K,sigmaPrice[[i+8]],NIntegrate[(Exp[x]-K)
f[x,muTilde[[i+8]],sigmaTilde[[i+8]]],x,Log[K], ∞],
i,1,92,1],K,KStart,KEnd,2],1];
```

After the numerical integration, we obtain 100 prices for each strike (total: 100×20 prices.) The output is too large to produce here (See CD) so we do not display. We plot the surface of the computed prices to see how sensitive they are to different strike price and stock price volatilities.

```
ListPlot3D[LNPrices, AxesLabel -> K,σ,"Price",LabelStyle->Directive[Large]]
```

The output is identical to Fig.7.1a.

To check accuracy, MC counterpart prices are needed. We simulated 10^6 trajectories to ensure accuracy with use of MATLAB. With the terminology introduced in the Monte Carlo Simulation chapter, we first wrote a function to simulate path dependent VWAP call options:

```
% a realisation of a squared Ornstein-Uhblenbeck process
%vwapou.m
```

```
T=1; %final time
N=500; %number of time steps
lambda=2;a=22;sig=5;%parameters
x=nan(N,1);%generate N by 1 not a Not-A-Number
x(1)=22;%initial value
h=T/(N-1); %step size
t=0:h:T; % time
f=@(z,x)(exp(-lambda*h)*x+sig*sqrt((1-exp(-2*h*lambda))/(2*lambda))*z);
for i=2:N
    x(i)=f(randn,x(i-1));
end
x=x+a*(1-exp(-lambda*t'));
u=x.^2;
subplot(2,1,1); plot(t,x)
subplot(2,1,2); plot(t,u)
```

```
%Simulate VWAP call with S_t dynamics is Gemetric Levy process
function [vwapcall,stdcall,vwapm1,stdm1,vwapm2,stdm2,vwapm3]
=vwap_rho_vg(S0,K,rho,sigvg,sigtil,nu,theta,lambda,a,sigou,X0,T,N,MSim)
tic;  % use tic toc to elapsed CPU time
%initialisation
dt=T/N;
C=1/nu;   % Assume C1=C2 for traceability
G=(sqrt(theta.^2*nu.^2/4+sigvg.^2*nu/2)-theta*nu/2).^(-1);
M=(sqrt(theta.^2*nu.^2/4+sigvg.^2*nu/2)+theta*nu/2).^(-1);
muX=a*(1-exp(-lambda*dt));
sigX2=(sigou^2/(2*lambda))*(1-exp(-2*lambda*dt));

mu_vg=0.1;
m=mu_vg+log(1-sigvg.^2*nu/2-theta*nu)/nu-0.5*sigtil.^2;
r=0;

sum1=0;
sum1a=0;
sum1b=0;
sum2=0;
sum3=0;
sum4=0;
%To generate stock path, we first generate jumps
for j=1:MSim

S_vg=S0;
Xti=X0;
Uti=X0.*X0;
SumUt=X0.*X0;
SumStUt=S_vg.*Uti;
for i=2:N+1
```

```
%First, simulate jumps
g1=gamrnd(dt*C,1/M);    % generate gamma random deviate
g2=gamrnd(dt*C,1/G);   % generate gamma random variate
TJS=0;
JS=g1-g2;
TJS=TJS+JS;
%Now as usual, generate path for St
BM=randn(2,1); %generate two N(0,1) random variate
gz=sigtil.*sqrt(dt)*(rho*BM(1)+sqrt(1-rho^2)*BM(2));
S_vg= S_vg.*exp(m*dt+gz+TJS);
%generate path for Volume
Xti=exp(-lambda*dt)*Xti+muX+sqrt(sigX2)*BM(1);
Uti=Xti.^2;
StUt=S_vg.*Uti;
% Accumulate values of denominator and numerator of the VWAP
SumUt=SumUt+Uti;
SumStUt=SumStUt+StUt;
end;
%Take the ratio we get VWAP
AT=SumStUt/SumUt;
sum1=sum1+AT;
%add to running sum
sum1a=sum1a+max(AT-K,0);
sum2=sum2+AT.^2;
sum3=sum3+AT.^3;
sum4=sum4+AT.^4;
end;
%discount back is not needed since r=0

%MC estimate of VWAP call
vwapcall=sum1a/MSim
stdcall=sqrt((sum1b/MSim-vwapcall.^2)/MSim)   %stderror of vwap call
%MC estimate of VWAP moment1
vwapm1=sum1/MSim
stdm1=sqrt((sum2/MSim-vwapm1.^2)/MSim)        %stderror of moment1
%MC estimate of VWAP moment1
vwapm2=sum2/MSim
stdm2=sqrt((sum4/MSim-vwapm2.^2)/MSim)        %stderror of moment2
vwapm3=sum3/MSim

toc
%Elapsed time is 46897.359567 seconds
%vwap_rho_vg(110,100,0.2,0.1,vol,0.1,-0.14,2,22,5,22,1,500,10)
%vwap_rho_vg(110,K,0.2,0.1,vol,0.1,-0.14,2,22,5,22,1,500,10) %for surface plot
```

One great advantage of MATLAB is the capability of handling vectors and matrices, vectored valued can be used directly. Hence, to compute the 100 prices at each strike, we set $\sigma = [0.01 : 0.01 : 1]$,

$K = [100 : 1 : 120]$. then with use of the f function, the previous mentioned *vwap_rho_vg* function was implemented in obtained the desired quantities of interest. Nevertheless, the codes are the following.

```
% MC option prices surface plot code
T=1;r=0;
K=[100:1:120];
S0=110;
vol=[0.01:0.01:1];
[X,Y]=meshgrid(K,vol);
f=@(K,vol) vwap_rho_vg(110,K,0.5,0.1,vol,0.1,-0.14,2,22,5,22,1,500,1000000)
[call,sdc,m1,sd1,m2,sd2,m3]=f(X,Y); %stored all
%xlswrite('VWAP_GL_MC_rho0.xls',[call,sdc,m1,sd1,m2,sd2,m3]) %output data to excel
```

One can also plot the surface via the $surf()$ function to visualise how prices varies across volatilities and strikes. The output was presented previously as in Fig.7.2.

Now we check the accuracy of the lognormal approximation at three level of strike prices $K = 100, 110, 120$. First, import the Monte Carlo option prices and store them in a table

```
mcPriceKcsv=Import["F:\\Thesis codes\\FinalMoment\\VWAP_GL_MC_rho0.csv","CSV"];
Table[mcPriceKcsv[[i]][[1]],{i,1,Length[mcPriceKcsv]}];
mcCallK100=Table[mcPriceKcsv[[i]][[2]],{i,9,Length[mcPriceKcsv]}];
mcCallK110=Table[mcPriceKcsv[[i]][[12]],{i,9,Length[mcPriceKcsv]}];
mcCallK120=Table[mcPriceKcsv[[i]][[22]],{i,9,Length[mcPriceKcsv]}];
```

Copy the previous computed analytical prices data at the desired strike level, store them in a table as well.

```
LNCallK100 = Table[LNPriceK100[[i]][[3]], {i, 1, Length[LNPriceK100], 1}]
(*Table of call price struck at 100 i from 1  to 100 in step of 1*)

LNCallK110 = Table[LNPriceK110[[i]][[3]], {i, 1, Length[LNPriceK110], 1}]
(*Table of call price struck at 100 i from 1  to 100 in step of 1*)

LNCallK120 = Table[LNPriceK120[[i]][[3]], {i, 1, Length[LNPriceK120], 1}]
(*Table of call price struck at 120 i from 1  to 100 in step of 1*)
```

Compute the relative error, and plot them.

```
relativeK100
=Table[{LNPriceK100[[i]][[2]],Abs[mcCallK100[[i]]-LNCallK100[[i]]]/mcCallK100[[i]]*100},
{i,1,92,1}]

ListLinePlot[relativeK100,PlotStyle->{Thick},AxesLabel->{\[Sigma],Text["Relative Error (%)"]},
PlotRange->{{0.09,1}}]

relativeK110
=Table[{LNPriceK110[[i]][[2]],Abs[mcCallK110[[i]]-LNCallK110[[i]]]/mcCallK110[[i]]*100},
{i,1,92,1}]

ListLinePlot[relativeK110,PlotStyle->{Thick},AxesLabel->{\[Sigma],Text["Relative Error (%)"]},
```

```
PlotRange->{{0.09,1}}]

relativeK120
=Table[{LNPriceK120[[i]][[2]],Abs[mcCallK120[[i]]-LNCallK120[[i]]]/mcCallK120[[i]]*100},
{i,1,92 ,1}]

ListLinePlot[relativeK120,PlotStyle->{Thick},AxesLabel->{\[Sigma],Text["Relative Error (%)"]},
PlotRange->{{0.09,1}}]
```

Now to compute the GIG prices, open the *GIGprice_rho0.nb*, import the moments data as usual,

```
csvFile=Import["F:\\Thesis codes\\FinalMoment\\NEW\\GL 2nd moment 100_rho0.csv","CSV"];

sigmaPrice=Table[csvFile[[i]][[1]], {i,2,Length[csvFile]}];
mu1=Table[csvFile[[i]][[2]], {i,2,Length[csvFile]}];
mu2=Table[csvFile[[i]][[3]], {i,2,Length[csvFile]}];
mu3=Table[csvFile[[i]][[4]], {i,2,Length[csvFile]}];
R1[p_,u_]:=(BesselK[p+2,u]BesselK[p,u])/BesselK[p+1,u]
R2[p_,u_]:=(BesselK[p+3,u](BesselK[p,u])^2)/(BesselK[p+1,u])^3
```

Use the **FindRoot** command to find p and u, Notice that Bessel$K[p,u]$ implemented in Mathematica gives us the modified Bessel function of the second kind $K_p(u)$, here p is the order of the Bessel function evaluated at point u

```
puRoots = Table[FindRoot[{R1[p,u]==mu2[[i]]/(mu1[[i]])^2,R2[p,u]==mu3[[i]]/(mu1[[i]])^3},
{{p,-1.5},{u,0.5}}], {i,1,Length[mu1],1}]
```

And the following is obtained.

```
FindRoot::lstol: The line search decreased the step size to within tolerance specified
by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease
in the merit function.
 You may need more than MachinePrecision digits of working precision to meet
these tolerances. >>

{{p->-213.252,u->139.272},{p->-207.933,u->135.755},{p->-199.615,u->130.253},
{p->-189.023,u->123.249},{p->-176.952,u->115.267},{p->-164.149,u->106.804},
{p->-151.231,u->98.2653},{p->-138.657,u->89.9567},{p->-126.735,u->82.0804},
{p->-115.642,u->74.7546},{p->-105.46,u->68.0333},{p->-96.2052,u->61.9255},
{p->-87.8465,u->56.4117},{p->-80.3293,u->51.4553},{p->-73.5858,u->47.0111},
{p->-67.5434,u->43.0311},{p->-62.1304,u->39.4676},{p->-57.2787,u->36.2756},
{p->-52.9257,u->33.4134},{p->-49.0143,u->30.8433},{p->-45.4939,u->28.5317},
{p->-42.3193,u->26.4487},{p->-39.4507,u->24.568},{p->-36.8534,u->22.8664},
{p->-34.4964,u->21.3236},{p->-32.3532,u->19.9219},{p->-30.4,u->18.6457},
{p->-28.6163,u->17.4813},{p->-26.9839,u->16.4168},{p->-25.4871,u->15.4416},
{p->-24.1118,u->14.5465},{p->-22.8458,u->13.7234},{p->-21.6781,u->12.965},
{p->-20.5992,u->12.2652},{p->-19.6005,u->11.6181},{p->-18.6746,u->11.0189},
{p->-17.8147,u->10.4631},{p->-17.0148,u->9.94674},{p->-16.2696,u->9.46632},
{p->-15.5743,u->9.01867},{p->-14.9246,u->8.60099},{p->-14.3168,u->8.21072},
```

```
{p->-13.7473,u->7.84559},{p->-13.213,u->7.50355},{p->-12.7112,u->7.18274},
{p->-12.2392,u->6.88149},{p->-11.7948,u->6.59828},{p->-11.3759,u->6.33174},
{p->-10.9805,u->6.08061},{p->-10.6071,u->5.84375},{p->-10.2539,u->5.62013},
{p->-9.91953,u->5.4088},{p->-9.60271,u->5.2089},{p->-9.30221,u->5.01963},
{p->-9.01692,u->4.84027},{p->-8.74584,u->4.67014},{p->-8.48802,u->4.50865},
{p->-8.24261,u->4.35522},{p->-8.00883,u->4.20933},{p->-7.78593,u->4.07051},
{p->-7.57324,u->3.93831},{p->-7.37014,u->3.81233},{p->-7.17604,u->3.69218},
{p->-6.99042,u->3.57752},{p->-6.81278,u->3.46801},{p->-6.64265,u->3.36336},
{p->-6.4796,u->3.26328},{p->-6.32323,u->3.16752},{p->-6.17317,u->3.07583},
{p->-6.02906,u->2.98798},{p->-5.8906,u->2.90376},{p->-5.75746,u->2.82297},
{p->-5.62937,u->2.74543},{p->-5.50606,u->2.67097},{p->-5.38729,u->2.59942},
{p->-5.27281,u->2.53064},{p->-5.1624,u->2.46447},{p->-5.05587,u->2.40079},
{p->-4.95302,u->2.33948},{p->-4.85366,u->2.2804},{p->-4.75763,u->2.22347},
{p->-4.66476,u->2.16856},{p->-4.5749,u->2.11558},{p->-4.48791,u->2.06444},
{p->-4.40365,u->2.01505},{p->-4.32199,u->1.96734},{p->-4.24281,u->1.92122},
{p->-4.16599,u->1.87661},{p->-4.09144,u->1.83346},{p->-4.01904,u->1.7917},
{p->-3.94869,u->1.75126},{p->-3.88031,u->1.71209},{p->-3.81381,u->1.67413},
{p->-3.7491,u->1.63733},{p->-3.6861,u->1.60164},{p->-3.62474,u->1.56701},
{p->-3.56494,u->1.5334},{p->-3.50665,u->1.50077},{p->-3.44978,u->1.46907},
{p->-3.39428,u->1.43828}}
```

Extract p and u from the list so that we can use them conveniently.

```
pRoots = Table[puRoots[[i]][[1]][[2]],{i,1,Length[puRoots],1}]
uRoots = Table[puRoots[[i]][[2]][[2]],{i,1,Length[puRoots],1}]

{-213.252,-207.933,-199.615,-189.023,-176.952,-164.149,-151.231,-138.657,-126.735,-115.642,
-105.46,-96.2052,-87.8465,-80.3293,-73.5858,-67.5434,-62.1304,-57.2787,-52.9257,-49.0143,
-45.4939,-42.3193,-39.4507,-36.8534,-34.4964,-32.3532,-30.4,-28.6163,-26.9839,-25.4871,
-24.1118,-22.8458,-21.6781,-20.5992,-19.6005,-18.6746,-17.8147,-17.0148,-16.2696,-15.5743,
-14.9246,-14.3168,-13.7473,-13.213,-12.7112,-12.2392,-11.7948,-11.3759,-10.9805,-10.6071,
-10.2539,-9.91953,-9.60271,-9.30221,-9.01692,-8.74584,-8.48802,-8.24261,-8.00883,-7.78593,
-7.57324,-7.37014,-7.17604,-6.99042,-6.81278,-6.64265,-6.4796,-6.32323,-6.17317,-6.02906,
-5.8906,-5.75746,-5.62937,-5.50606,-5.38729,-5.27281,-5.1624,-5.05587,-4.95302,-4.85366,
-4.75763,-4.66476,-4.5749,-4.48791,-4.40365,-4.32199,-4.24281,-4.16599,-4.09144,-4.01904,
-3.94869,-3.88031,-3.81381,-3.7491,-3.6861,-3.62474,-3.56494,-3.50665,-3.44978,-3.39428}


{139.272,135.755,130.253,123.249,115.267,106.804,98.2653,89.9567,82.0804,74.7546,68.0333,
61.9255,56.4117,51.4553,47.0111,43.0311,39.4676,36.2756,33.4134,30.8433,28.5317,26.4487,
24.568,22.8664,21.3236,19.9219,18.6457,17.4813,16.4168,15.4416,14.5465,13.7234,12.965,
12.2652,11.6181,11.0189,10.4631,9.94674,9.46632,9.01867,8.60099,8.21072,7.84559,7.50355,
7.18274,6.88149,6.59828,6.33174,6.08061,5.84375,5.62013,5.4088,5.2089,5.01963,4.84027,
4.67014,4.50865,4.35522,4.20933,4.07051,3.93831,3.81233,3.69218,3.57752,3.46801,3.36336,
3.26328,3.16752,3.07583,2.98798,2.90376,2.82297,2.74543,2.67097,2.59942,2.53064,2.46447,
2.40079,2.33948,2.2804,2.22347,2.16856,2.11558,2.06444,2.01505,1.96734,1.92122,1.87661,
1.83346,1.7917,1.75126,1.71209,1.67413,1.63733,1.60164,1.56701,1.5334,1.50077,1.46907,
1.43828}
```

Now use the p, u roots as inputs to find a, b roots via the **FindRoot** command.

```
abRoots=Table[FindRoot[{(b/a)^(1/2) BesselK[pRoots[[i]]+1,uRoots[[i]]]/BesselK[pRoots[[i]],
uRoots[[i]]]==mu1[[i]],
Sqrt[a b]==uRoots[[i]]},{{a,0.2},{b,20.2}}], {i,1,Length[puRoots],1}]
```

```
{{22->0.359611,0->53938.1},{22->0.350466,0->52585.3},{22->0.336163,0->50469.4},
{22->0.317953,0->47775.},{22->0.297208,0->44704.6},{22->0.275211,0->41448.2},
{22->0.253025,0->38162.5},{22->0.23144,0->34964.6},{22->0.210983,0->31932.3},
{22->0.191962,0->29111.3},{22->0.174515,0->26522.3},{22->0.158666,0->24168.9},
{22->0.144363,0->22043.5},{22->0.131512,0->20132.4},{22->0.119994,0->18418.},
{22->0.109683,0->16882.},{22->0.100456,0->15506.2},{22->0.0921958,0->14273.1},
{22->0.0847927,0->13166.9},{22->0.0781492,0->12173.},{22->0.0721776,0->11278.5},
{22->0.0668,0->10472.1},{22->0.061948,0->9743.45},{22->0.0575612,0->9083.77},
{22->0.0535868,0->8485.26},{22->0.0499786,0->7941.07},{22->0.0466961,0->7445.21},
{22->0.0437036,0->6992.46},{22->0.0409702,0->6578.19},{22->0.0384684,0->6198.37},
{22->0.0361743,0->5849.46},{22->0.0340666,0->5528.3},{22->0.0321267,0->5232.16},
{22->0.0303382,0->4958.58},{22->0.0286863,0->4705.4},{22->0.0271582,0->4470.7},
{22->0.0257423,0->4252.77},{22->0.0244284,0->4050.1},{22->0.0232073,0->3861.33},
{22->0.0220709,0->3685.24},{22->0.0210117,0->3520.74},{22->0.0200233,0->3366.87},
{22->0.0190997,0->3222.73},{22->0.0182356,0->3087.54},{22->0.0174262,0->2960.59},
{22->0.0166671,0->2841.22},{22->0.0159544,0->2728.85},{22->0.0152846,0->2622.96},
{22->0.0146544,0->2523.05},{22->0.0140608,0->2428.7},{22->0.0135012,0->2339.49},
{22->0.0129731,0->2255.06},{22->0.0124743,0->2175.09},{22->0.0120027,0->2099.25},
{22->0.0115565,0->2027.28},{22->0.0111339,0->1958.91},{22->0.0107333,0->1893.9},
{22->0.0103534,0->1832.05},{22->0.00999269,0->1773.14},{22->0.00965002,0->1717.},
{22->0.00932423,0->1663.44},{22->0.00901425,0->1612.32},{22->0.00871913,0->1563.48},
{22->0.00843795,0->1516.79},{22->0.00816988,0->1472.13},{22->0.00791413,0->1429.36},
{22->0.00767,0->1388.4},{22->0.0074368,0->1349.13},{22->0.0072139,0->1311.45},
{22->0.00700074,0->1275.29},{22->0.00679677,0->1240.56},{22->0.00660147,0->1207.18},
{22->0.00641438,0->1175.08},{22->0.00623505,0->1144.19},{22->0.00606308,0->1114.45},
{22->0.00589807,0->1085.8},{22->0.00573966,0->1058.19},{22->0.00558751,0->1031.55},
{22->0.00544131,0->1005.85},{22->0.00530075,0->981.039},{22->0.00516556,0->957.068},
{22->0.00503547,0->933.901},{22->0.00491024,0->911.497},{22->0.00478962,0->889.821},
{22->0.00467341,0->868.839},{22->0.00456139,0->848.517},{22->0.00445337,0->828.827},
{22->0.00434916,0->809.737},{22->0.0042486,0->791.222},{22->0.00415153,0->773.255},
{22->0.00405778,0->755.812},{22->0.00396721,0->738.869},{22->0.00387969,0->722.404},
{22->0.00379509,0->706.397},{22->0.00371329,0->690.828},{22->0.00363416,0->675.677},
{22->0.00355761,0->660.926},{22->0.00348352,0->646.559},{22->0.00341181,0->632.559},
{22->0.00334238,0->618.911}}
```

Extract a, b roots from the list so that we can make use of them

```
aRoots = Table[abRoots[[i]][[1]][[2]],{i,1,Length[puRoots],1}]
bRoots = Table[abRoots[[i]][[2]][[2]],{i,1,Length[puRoots],1}]
```

```
{0.359611,0.350466,0.336163,0.317953,0.297208,0.275211,0.253025,0.23144,0.210983,
```

```
0.191962,0.174515,0.158666,0.144363,0.131512,0.119994,0.109683,0.100456,0.0921958,
0.0847927,0.0781492,0.0721776,0.0668,0.061948,0.0575612,0.0535868,0.0499786,0.0466961,
0.0437036,0.0409702,0.0384684,0.0361743,0.0340666,0.0321267,0.0303382,0.0286863,0.0271582,
0.0257423,0.0244284,0.0232073,0.0220709,0.0210117,0.0200233,0.0190997,0.0182356,0.0174262,
0.0166671,0.0159544,0.0152846,0.0146544,0.0140608,0.0135012,0.0129731,0.0124743,0.0120027,
0.0115565,0.0111339,0.0107333,0.0103534,0.00999269,0.00965002,0.00932423,0.00901425,
0.00871913,0.00843795,0.00816988,0.00791413,0.00767,0.0074368,0.0072139,0.00700074,
0.00679677,0.00660147,0.00641438,0.00623505,0.00606308,0.00589807,0.00573966,0.00558751,
0.00544131,0.00530075,0.00516556,0.00503547,0.00491024,0.00478962,0.00467341,0.00456139,
0.00445337,0.00434916,0.0042486,0.00415153,0.00405778,0.00396721,0.00387969,0.00379509,
0.00371329,0.00363416,0.00355761,0.00348352,0.00341181,0.00334238}
```

```
{53938.1,52585.3,50469.4,47775.,44704.6,41448.2,38162.5,34964.6,31932.3,29111.3,
26522.3,24168.9,22043.5,20132.4,18418.,16882.,15506.2,14273.1,13166.9,12173.,
11278.5,10472.1,9743.45,9083.77,8485.26,7941.07,7445.21,6992.46,6578.19,6198.37,
5849.46,5528.3,5232.16,4958.58,4705.4,4470.7,4252.77,4050.1,3861.33,3685.24,
3520.74,3366.87,3222.73,3087.54,2960.59,2841.22,2728.85,2622.96,2523.05,2428.7,
2339.49,2255.06,2175.09,2099.25,2027.28,1958.91,1893.9,1832.05,1773.14,1717.,
1663.44,1612.32,1563.48,1516.79,1472.13,1429.36,1388.4,1349.13,1311.45,1275.29,1240.56,
1207.18,1175.08,1144.19,1114.45,1085.8,1058.19,1031.55,1005.85,981.039,957.068,
933.901,911.497,889.821,868.839,848.517,828.827,809.737,791.222,773.255,755.812,
738.869,722.404,706.397,690.828,675.677,660.926,646.559,632.559,618.911}
```

All roots are real. Now using the GIG distribution as the state price density, the prices could be computed by numerical integration via NIntegrate.

$$f[\text{x\_},\text{p\_},\text{a\_},\text{b\_}]:=\frac{(a/b)^{p/2}}{2\text{BesselK}\left[p,\sqrt{ab}\right]}x^{(p-1)}\text{Exp}\left[-\left(ax+\frac{b}{x}\right)/2\right]$$

KStart = 100;
KEnd = 120;

```
GIGPrices=
Flatten[Table[Table[Strike, sigmaPrice[[i]],
NIntegrate[(x-Strike)f[x,pRoots[[i]],aRoots[[i]],bRoots[[i]]],
x,Strike, ∞],
i,1,Length[puRoots],1],
Strike,KStart,KEnd,2] ,1]
```

Again, output too large so not produce here (See CD).

Plot the surface to see how price evolves across different strike levels and volatilities.

```
ListPlot3D[GIGPrices, AxesLabel -> K,σ,"   Price",LabelStyle->Directive[Large]]
```

The output is identical to Fig.7.1b. To check accuracy, the exact procedure was followed as in the *LNprice_rho0.nb*. (See CD)

Appendix C

All other MATLAB and Mathematica codes

C.1 MATLAB codes for the plots of trajectories and GBM model

1. MATLAB codes for chapter 4

```
%10 trajectories of GL with VG jumps are plotted with use of Euler scheme
clear all;
 increment = 500; %500 discretisation
 ntraj = 10;
 T = 1;                    % Maturity
 S0 = 110;      % Initial stock price
 %Model parameter
 sigmaBM = 0.10; % volatility
 sigmaVG=0.3;
 theta= -0.14;
 nu=0.2;
 mu_vg=0.1;
 m=mu_vg+log(1-sigmaVG^2*nu/2-theta*nu)/nu-0.5*sigmaBM^2;
 C=1/nu;
 G=(sqrt(theta^2*nu^2/4+sigmaVG^2*nu/2)-theta*nu/2)^(-1);
 M=(sqrt(theta^2*nu^2/4+sigmaVG^2*nu/2)+theta*nu/2)^(-1);
 dt = T/increment; % time step
 c1=m*dt;
 c2=sigmaBM*sqrt(dt);

 T=1;

 for i = 1:increment+1
    t(i) = (i-1)*dt;
 end
```

```
 for i = 1 : ntraj
% initialize stock price for each simulation
 S(1,i) = S0;


    for j = 2:increment+1
        variate = normrnd(0,1); % generate gaussian random deviate
        g1=gamrnd(dt*C,1/M);    % generate gamma random deviate
        g2=gamrnd(dt*C,1/G)   ;% generate gamma random deviate
        TJS = 0;
        JS = g1-g2;
        TJS = TJS + JS;
        S(j,i) = S(j-1,i)*exp( c1 +c2*deviate+ TJS );
    end
 end
plot(t,S)
```

```
%10 trajectories of GBM are plotted with use of Euler scheme
clear all;
 increment = 500; %500 discretisation
 ntraj = 10; %number of trajectories
 T = 1;                     % Maturity
 S0 = 110;  % Initial stock price
 sigma = 0.10; % volatility
 mu=0.1
 dt = T/increment; % time step
 c1=(mu-0.5*sigma^2)*dt
 c2=sigma*sqrt(dt);
 T=1;
 for i = 1:increment+1
    t(i) = (i-1)*dt;
 end
 for i = 1 : ntraj
% initialize stock price for each simulation
 S(1,i) = S0;
    for j = 2:increment+1
        deviate = normrnd(0,1);
        S(j,i) = S(j-1,i)*exp( c1 +c2*deviate );
    end
 end
plot(t,S)
```

```
% one trajectory of a squared Ornstein-Uhblenbeck process
%vwapou.m
T=1; %final time
N=500; %number of time steps
lambda=2;a=22;sig=5;%parameters
```

```
x=nan(N,1);%generate N by 1 not a Not-A-Number
x(1)=22;%initial value
h=T/(N-1); %step size
t=0:h:T; % time
f=@(z,x)(exp(-lambda*h)*x+sig*sqrt((1-exp(-2*h*lambda))/(2*lambda))*z);
for i=2:N
    x(i)=f(randn,x(i-1));
end
x=x+a*(1-exp(-lambda*t'));
u=x.^2;
subplot(2,1,1); plot(t,x)
subplot(2,1,2); plot(t,u)

%Simulate VWAP call with S_t dynamics is Gemetric Brownian motion process
function [vwapcall,stdcall,vwapm1,stdm1,vwapm2,stdm2,vwapm3]
= vwap_rho_gbm( S0,K,rho,mu,sigma,lambda,a,sigou,X0,T,N,MSim )
tic;
dt=T/N;
muX=a*(1-exp(-lambda*dt));
sigX2=(sigou^2/(2*lambda))*(1-exp(-2*lambda*dt));
r=0;

%initialisation
sum1=0;
sum1a=0;
sum1b=0;
sum2=0;
sum3=0;
sum4=0;
for j=1:MSim
S_gbm=S0;
Xti=X0;
Uti=0;
SumUt=X0.*X0;
SumStUt=S_gbm.*Uti;
for i=2:N+1

BM=randn(2,1);
S_gbm=S_gbm.*exp((mu-0.5*sigma.^2)*dt
+sigma*sqrt(dt).*(rho*BM(1)+sqrt(1-rho^2)*BM(2)));
Xti=exp(-lambda*dt)*Xti+muX+sqrt(sigX2)*BM(1);
Uti=Xti.^2;
StUt=S_gbm.*Uti;
SumUt=SumUt+Uti;
SumStUt=SumStUt+StUt;
end;
AT=SumStUt/SumUt;
```

```
sum1=sum1+AT;
sum1a=sum1a+max(AT-K,0);
sum1b=sum1b+max(AT-K,0).^2;


sum2=sum2+AT.^2;
sum3=sum3+AT.^3;
sum4=sum4+AT.^4;
end;
%discount back is not needed since r=0

vwapcall=sum1a/MSim
stdcall=sqrt((sum1b/MSim-vwapcall.^2)/MSim)
vwapm1=sum1/MSim
stdm1=sqrt((sum2/MSim-vwapm1.^2)/MSim)
vwapm2=sum2/MSim
stdm2=sqrt((sum4/MSim-vwapm2.^2)/MSim)
vwapm3=sum3/MSim

toc
%Elapsed time is 288835.439527 seconds
%[call,sdc,m1,sd1,m2,sd2,m3]=vwap_rho_gbm( 110,100,0.1,0.1,0.1,2,22,5,22,1,500,1000)
%[call,sdc,m1,sd1,m2,sd2,m3]=vwap_rho_gbm( 110,K,0.1,0.1,vol,2,22,5,22,1,500,1000)%for surface plot
```

C.2 Analytical approximation under the GBM model

The following are the analogous Mathematica codes implement our analytical formulae under the classical GBM model. Notebook GBM 2nd moment *basic_rho0.nb* compute first and second moments given scalar parameter values

%S0=110; σ=5; T=1; λ=2; b=0; ρ=0; σVG=0.1; σBM=0.1; σRHO=σBM* $\sqrt{1-\rho\hat{}2}$;

$$m[\mathrm{t\_}] = a e^{-\lambda t} + a\left(1-e^{-\lambda t}\right);$$

Needs["MultivariateStatistics"]

$$\mathrm{PDF}\left[\mathrm{MultinormalDistribution}\left[\left\{\mu\mathrm{x1}+\frac{\sigma 13}{\sigma 33}(z-\mu\mathrm{x3}),\mu\mathrm{x2}+\frac{\sigma 23}{\sigma 33}(z-\mu\mathrm{x3})\right\},\right.\right.$$
$$\left.\left.\left\{\left\{\sigma 11-\frac{(\sigma 13)^2}{\sigma 33},\sigma 12-\frac{\sigma 13\sigma 23}{\sigma 33}\right\},\left\{\sigma 12-\frac{\sigma 13\sigma 23}{\sigma 33},\sigma 22-\frac{(\sigma 23)^2}{\sigma 33}\right\}\right\}\right],\{x,y\}\right];$$

$$\mathrm{expon}[\mathrm{x\_},\mathrm{y\_}] = \frac{1}{2}\left(-\left(\frac{\left(y-\mu\mathrm{x2}-\frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\left(-\sigma 12+\frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.\right.$$
$$\left.+\frac{\left(x-\mu\mathrm{x1}-\frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(\sigma 22-\frac{\sigma 23^2}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(x-\mu\mathrm{x1}-\frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)$$
$$-\left(\frac{\left(\sigma 11-\frac{\sigma 13^2}{\sigma 33}\right)\left(y-\mu\mathrm{x2}-\frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.$$
$$\left.\left.+\frac{\left(x-\mu\mathrm{x1}-\frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(-\sigma 12+\frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(y-\mu\mathrm{x2}-\frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\right)$$
$$-2q\sigma x+\frac{(\kappa-\lambda)y^2}{2}; \text{(*!!!!!!!!!!!!!!!!!!!!!!!!!!*)}$$

$$G = \mathrm{FullSimplify}[\mathrm{expon}[0,0]];$$

F = FullSimplify[Coefficient[expon[x, y], xy]];

A = −FullSimplify [Coefficient [expon[x, 0], x^2]] ;

B = FullSimplify [Coefficient [expon[0, y], y^2]] ;

Cx = FullSimplify[Coefficient[expon[x, 0], x]];

Dy = FullSimplify[Coefficient[expon[0, y], y]];

JYt = FullSimplify $\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G, z\right]\right]$;

HYt2 = FullSimplify $\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G, z^2\right]\right]$;

constZ[z_] = $\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G$;

L = FullSimplify[constZ[0]];

Check that the coefficient H, J, L are correct

FullSimplify $\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G-\text{HYt2}*z^2-\text{JYt}*z-L\right]$

0

Check that the coefficient A, B, C, D, F are correct

FullSimplify $\left[\text{expon}[x,y]-\left((\text{Cx}*x)+(\text{Dy}*y)+(F*(xy))+\left(-A*x^2\right)+\left(B*y^2\right)+G\right)\right]$

0

OC1 = FullSimplify $\left[\frac{2\pi}{\sqrt{A}\sqrt{-\frac{4AB+F^2}{A}}}/\left(2\pi\sqrt{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\right]$;

Now the outer integral

*)

PDF[MultinormalDistribution[{μa, μb}, {{σaa, σab}, {σab, σbb}}], {x, y}];

expon1[x_, y_] = $\frac{1}{2}\left(-(y-\mu\text{b})\left(\frac{(y-\mu\text{b})\sigma\text{aa}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}-\frac{(x-\mu\text{a})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)-(x-\mu\text{a})\left(-\frac{(y-\mu\text{b})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}+\frac{(x-\mu\text{a})\sigma\text{bb}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right)$
+(c1)x + (c2)x^2 + (c3)y;

G1 = FullSimplify[expon1[0, 0]];

F1 = FullSimplify[Coefficient[expon1[x, y], xy]];

A1 = −FullSimplify [Coefficient [expon1[x, 0], x^2]] ;

B1 = FullSimplify [Coefficient [expon1[0, y], y^2]] ;

Cx1 = FullSimplify[Coefficient[expon1[x, 0], x]];

Dy1 = FullSimplify[Coefficient[expon1[0, y], y]];

(*CheckthatthecoefficientsA1, B1, C1, D1, F1andG1arecorrect*)

FullSimplify $\left[\text{expon1}[x,y]-\left((\text{Cx1}*x)+(\text{Dy1}*y)+(\text{F1}*(xy))+\left(-\text{A1}*x^2\right)+\left(\text{B1}*y^2\right)+\text{G1}\right)\right]$

0

OC2 = FullSimplify $\left[\frac{2\pi}{\sqrt{\text{A1}}\sqrt{-\frac{4\text{A1B1}+\text{F1}^2}{\text{A1}}}}/\left(2\pi\sqrt{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right]$;

```
(*Now need to collect coefficients for the triple integral*)
exponTrip[u_, v_, w_] = 1/2 (-w (w(-σab^2+σaaσbb)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)
+ v(σabσac-σaaσbc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)
+ u(-σacσbb+σabσbc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc))
-v (w(σabσac-σaaσbc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)
+ v(-σac^2+σaaσcc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)
+ u(σacσbc-σabσcc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc))
-u (w(-σacσbb+σabσbc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)
+ v(σacσbc-σabσcc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)
+ u(-σbc^2+σbbσcc)/(-σac^2σbb+2σabσacσbc-σaaσbc^2-σab^2σcc+σaaσbbσcc)))
+(-2σm[t]z + J)u + (-zσ^2 + H) u^2 - 2qσv + (-2σ m[s]r)w + (-rσ^2) w^2;

C00 = FullSimplify[exponTrip[0, 0, 0]];

Cu2 = -Coefficient [exponTrip[u, 0, 0], u^2] ;

Cv2 = Coefficient [exponTrip[0, v, 0], v^2] ;

Cw2 = Coefficient [exponTrip[0, 0, w], w^2] ;

Cu = Coefficient[exponTrip[u, 0, 0], u];

Cv = Coefficient[exponTrip[0, v, 0], v];

Cw = Coefficient[exponTrip[0, 0, w], w];

Cuv = Coefficient[exponTrip[u, v, 0], uv];

Cuw = Coefficient[exponTrip[u, 0, w], uw];

Cvw = Coefficient[exponTrip[0, v, w], vw];

FullSimplify [exponTrip[u, v, w] - (-Cu2 * u^2 + Cv2 * v^2 + Cw2 * w^2 + Cu * u + Cv * v
+Cw * w + Cuv * (u * v) + Cuw * (u * w) + Cvw * (v * w) + C00)]

0


κ = Sqrt[λ^2 + 2qσ^2];

J = JYt;

H = HYt2;

c1 = -2zm[t]σ + J; (*!!!!!!!!!!!!!!!!!!!!!*)

c2 = -zσ^2 + H;

c3 = -2qσ;

gamma[z_, q_] = OC1 * OC2 * Exp[L] * Exp [(B1(Cx1)^2-Dy1(A1Dy1+Cx1 F1))/(4*(A1B1)+F1^2) + G1] ;
```

(*Now to compute all the covariance*)

$\sigma 22 = \frac{1}{2\kappa}\left(1 - e^{-2\kappa T}\right);$

$\sigma 33 = \frac{1}{2\kappa}\left(1 - e^{-2\kappa t}\right);$

$\sigma 12 = \int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+T)}\left(e^{2\kappa s} - 1\right)\right)ds;$

$\sigma 13 = \int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa t} - 1\right)\right)ds;$

$\sigma 23 = \frac{1}{2\kappa}\left(e^{\kappa(t-T)} - e^{-\kappa(t+T)}\right);$

$\sigma \mathrm{aa} = \frac{1}{2\kappa}\left(1 - e^{-2\kappa t}\right);$

$\sigma \mathrm{ab} = \int_0^t m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s} - 1\right)\right)ds;$

(*σ11 and σbb were precomputed*)

$\sigma 11 = -\frac{a^2\left(2+e^{-2t\kappa}-2e^{(t-T)\kappa}+e^{-2T\kappa}-2e^{-(t+T)\kappa}+2t\kappa-2T\kappa\right)}{2\kappa^3};$

$\sigma \mathrm{bb} = \frac{a^2 e^{-2t\kappa}\left(-1+4e^{t\kappa}+e^{2t\kappa}(-3+2t\kappa)\right)}{2\kappa^3};$

$\sigma \mathrm{cc} = \frac{1}{2\kappa}\left(1 - e^{-2\kappa s}\right);$

$\sigma \mathrm{ac} = \frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s} - 1\right);$ (* s ≤ t*)

$\sigma \mathrm{bc} = -\frac{a e^{-(s+t)\kappa}\left(-1+e^{s\kappa}\right)\left(1+e^{s\kappa}-2e^{t\kappa}\right)}{2\kappa^2};$

Clear[S0, n, g, σVG, σBM, θ, v, μ];

ψ[n_]:=$\left(n * g + \frac{1}{2}n\text{^}2\sigma\mathrm{BM}\text{^}2\right)$

cf[n_]:=Exp[$t * \psi[n]$]

S0 = 110;

σ = 5;

T = 1;

λ = 2;

b = 0;

a = 22;

ρ = 0;σBM = 0.1; (*this is the diffusion coefficient in the vg stock price model*)

σRHO = σBM * $\left(\sqrt{1 - \rho\text{^}2}\right)$;

μ = 0.1;

g = μ − 0.5 * σBM^2;

μx1 = 0;

μx2 = 0;

μx3 = 0;

μa = 0;

μb = 0;

(*ψ[1];*)

Phi[z_, q_] = Exp$\left[-z(m[t])^2 - q\int_0^T (m[s])^2\,ds + \frac{(\lambda-\kappa)T}{2} - z*b - q*b*T\right]$ * gamma[z,q];

dzPhi[q_, t_] = Derivative[1, 0][Phi][z,q]/.$z \to 0$;

moment1 = NIntegrate[−S0 ∗ cf[1]Exp[0.5 ∗ ρ^2 ∗ σBM^2 ∗ t]dzPhi[q,t], {q, 0, ∞}, {t, 0, T}]

TimeUsed[]

115.681

13.353

OC2trip = $\frac{1}{\sqrt{\mathrm{Cu2}}\sqrt{-\frac{\mathrm{Cuv}^2+4\mathrm{Cu2Cv2}}{\mathrm{Cu2}}}}2\pi\Big/$

$\left(\sqrt{\frac{-\mathrm{Cuw}^2\mathrm{Cv2}+\mathrm{CuvCuwCvw}+\mathrm{Cu2Cvw}^2-\mathrm{Cuv}^2\mathrm{Cw2}-4\mathrm{Cu2Cv2Cw2}}{\mathrm{Cuv}^2\pi+4\mathrm{Cu2Cv2}\pi}}\right)\Big/$

$\left(2\sqrt{2}\pi^{3/2}\sqrt{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\right)$;

gamma2[z_, r_, q_] = OC1 ∗ OC2trip ∗ Exp[L] ∗ Exp$\Big[-\Big(\mathrm{Cuw}^2\left(\mathrm{Cv}^2-4\mathrm{C00Cv2}\right)+4\mathrm{C00Cu2Cvw}^2$

$-4\mathrm{Cu2CvCvwCw}+\mathrm{Cuv}^2\mathrm{Cw}^2$

$+4\mathrm{Cu2Cv2Cw}^2+\mathrm{Cuw}(4\mathrm{C00CuvCvw}-2\mathrm{CuCvCvw}-2\mathrm{CuvCvCw}+4\mathrm{CuCv2Cw})$

$-4\mathrm{C00Cuv}^2\mathrm{Cw2}+4\mathrm{Cu2Cv}^2\mathrm{Cw2}$

$-16\mathrm{C00Cu2Cv2Cw2}+\mathrm{CuCuv}(-2\mathrm{CvwCw}+4\mathrm{CvCw2})+\mathrm{Cu}^2\left(\mathrm{Cvw}^2-4\mathrm{Cv2Cw2}\right)\Big)\Big/$

$\left(4\left(\mathrm{Cuw}^2\mathrm{Cv2}-\mathrm{CuvCuwCvw}-\mathrm{Cu2Cvw}^2+\mathrm{Cuv}^2\mathrm{Cw2}+4\mathrm{Cu2Cv2Cw2}\right)\right)\Big]$;

Phi2[z_, r_, q_] = Exp$\left[-z(m[t])^2 - r(m[s])^2 - q\int_0^T (m[s])^2\,ds + \frac{(\lambda-\kappa)T}{2}\right]$ ∗ gamma2[z,r,q];

dzdrPhi2[q_, t_, s_] = ((Derivative[1, 1, 0][Phi2][z,r,q]/.$z \to 0$)/.$r \to 0$);

moment2 =

NIntegrate$\big[$S0^2qExp[s ∗ ψ[2] + ($t - s$) ∗ ψ[1] + 0.5 ∗ ρ^2 ∗ σBM^2 ∗ ($s + t$)]dzdrPhi2[q,t,s],

{q, 0, ∞}, {t, 0, T}, {s, 0, t}]+

NIntegrate$\big[$S0^2qExp[t ∗ ψ[2] + ($s - t$) ∗ ψ[1] + 0.5 ∗ ρ^2 ∗ σBM^2 ∗ ($s + t$)]dzdrPhi2[q,s,t],

{q, 0, ∞}, {t, 0, T}, {s, t, T}];

TimeUsed[]

Print[“First moment = ” <> ToString[moment1]]

Print[“Second moment = ” <> ToString[moment2]]

Print[“Variance = ” <> ToString[moment2 − moment1^2]]

Print[“SD = ” <> ToString[Sqrt[moment2 − moment1^2]]]

Print[“mu tilde = ” <> ToString[Log[moment1/S0]/T]]

Print$\Big[$“sig tilde = ” <> ToString$\Big[$Sqrt$\Big[$Log$\left[\frac{(\mathrm{moment2}-\mathrm{moment1\hat{}2})+(\mathrm{moment1})^2}{(\mathrm{moment1})^2}\right]\Big/T\Big]\Big]\Big]$

30.342

First moment = 115.681
Second moment = 13427.9
Variance = 45.934
SD = 6.77746
mu tilde = 0.0503521
sig tilde = 0.0585376 It worth mentioning the above results exactly match the original work in Novikov et al. Notice that moment 1 is invariant against the choice of correlation and pricing model. A simple reason is that the terms in the characteristic exponent cancelled out.
The *GBM 2nd moment 100_rho.nb* was coded in exactly the same manner given parameter values

```
S0=110; σ=5; T=1; λ=2; b=0; ρ=0; σVG=0.1; σBM=0.01,0.02,...,1; σRHO=σBM*√(1-ρ^2);
```

```
sigmaPrices = {0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2,
0.21, 0.22, 0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31,
0.32, 0.33, 0.34, 0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42,
0.43, 0.44, 0.45, 0.46, 0.47, 0.48, 0.49, 0.5, 0.51, 0.52, 0.53,
0.54, 0.55, 0.56, 0.57, 0.58, 0.59, 0.6, 0.61, 0.62, 0.63, 0.64,
0.65, 0.66, 0.67, 0.68, 0.69, 0.7, 0.71, 0.72, 0.73, 0.74, 0.75,
0.76, 0.77, 0.78, 0.79, 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86,
0.87, 0.88, 0.89, 0.9, 0.91, 0.92, 0.93, 0.94, 0.95, 0.96, 0.97,
0.98, 0.99, 1};
```

$$m[\mathrm{t\_}] = a e^{-\lambda t} + a\left(1 - e^{-\lambda t}\right);$$

Needs["MultivariateStatistics"]

$$\mathrm{PDF}\Big[\mathrm{MultinormalDistribution}\Big[\Big\{\mu\mathrm{x1} + \frac{\sigma 13}{\sigma 33}(z - \mu\mathrm{x3}), \mu\mathrm{x2} + \frac{\sigma 23}{\sigma 33}(z - \mu\mathrm{x3})\Big\}, \Big\{\Big\{\sigma 11 - \frac{(\sigma 13)^2}{\sigma 33}, \sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}\Big\}, \Big\{\sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}, \sigma 22 - \frac{(\sigma 23)^2}{\sigma 33}\Big\}\Big\}\Big], \{x, y\}\Big];$$

expon[x_, y_] =

$$\frac{1}{2}\left(-\left(\frac{\left(y-\mu\mathrm{x2}-\frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\left(-\sigma 12+\frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}} + \frac{\left(x-\mu\mathrm{x1}-\frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(\sigma 22-\frac{\sigma 23^2}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(x-\mu\mathrm{x1}-\frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right) - \left(\frac{\left(\sigma 11-\frac{\sigma 13^2}{\sigma 33}\right)\left(y-\mu\mathrm{x2}-\frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}} + \frac{\left(x-\mu\mathrm{x1}-\frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(-\sigma 12+\frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(y-\mu\mathrm{x2}-\frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\right)$$

$$-2q\sigma x + \frac{(\kappa-\lambda)y^2}{2};$$ (*!!!!!!!!!!!!!!!!!!!!!!!!!!*)

```
G = FullSimplify[expon[0, 0]];
F = FullSimplify[Coefficient[expon[x, y], xy]];
A = -FullSimplify[Coefficient[expon[x, 0], x^2]];
B = FullSimplify[Coefficient[expon[0, y], y^2]];
Cx = FullSimplify[Coefficient[expon[x, 0], x]];
Dy = FullSimplify[Coefficient[expon[0, y], y]];
```

$\text{JYt} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G,z\right]\right];$

$\text{HYt2} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G,z^2\right]\right];$

$\text{constZ}[\text{z\_}] = \frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G;$

$L = \text{FullSimplify}[\text{constZ}[0]];$

Check that the coefficients H, J and L are correct

$\text{FullSimplify}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G-\text{HYt2}*z^2-\text{JYt}*z-L\right]$

0

(*Check that the coefficients A,B,C,D,F and G are correct*)

$\text{FullSimplify}\left[\text{expon}[x,y]-\left((\text{Cx}*x)+(\text{Dy}*y)+(F*(xy))+\left(-A*x^2\right)+\left(B*y^2\right)+G\right)\right]$

0

$\text{OC1} = \text{FullSimplify}\left[\frac{2\pi}{\sqrt{A}\sqrt{-\frac{4AB+F^2}{A}}}/\left(2\pi\sqrt{-\sigma12^2+\sigma11\sigma22-\frac{\sigma13^2\sigma22}{\sigma33}+\frac{2\sigma12\sigma13\sigma23}{\sigma33}-\frac{\sigma11\sigma23^2}{\sigma33}}\right)\right];$

(*Now the outer integral*)

*)

$\text{PDF}[\text{MultinormalDistribution}[\{\mu\text{a},\mu\text{b}\},\{\{\sigma\text{aa},\sigma\text{ab}\},\{\sigma\text{ab},\sigma\text{bb}\}\}],\{x,y\}];$

$\text{expon1}[\text{x\_},\text{y\_}] = \frac{1}{2}\left(-(y-\mu\text{b})\left(\frac{(y-\mu\text{b})\sigma\text{aa}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}-\frac{(x-\mu\text{a})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)-(x-\mu\text{a})\left(-\frac{(y-\mu\text{b})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}+\frac{(x-\mu\text{a})\sigma\text{bb}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right)$
$+(\text{c1})x+(\text{c2})x^2+(\text{c3})y;$

$\text{G1} = \text{FullSimplify}[\text{expon1}[0,0]];$

$\text{F1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,y],xy]];$

$\text{A1} = -\text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[x,0],x^2\right]\right];$

$\text{B1} = \text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[0,y],y^2\right]\right];$

$\text{Cx1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,0],x]];$

$\text{Dy1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[0,y],y]];$

$\text{FullSimplify}\left[\text{expon1}[x,y]-\left((\text{Cx1}*x)+(\text{Dy1}*y)+(\text{F1}*(xy))+\left(-\text{A1}*x^2\right)+\left(\text{B1}*y^2\right)+\text{G1}\right)\right];$

$\text{OC2} = \text{FullSimplify}\left[\frac{2\pi}{\sqrt{\text{A1}}\sqrt{-\frac{4\text{A1B1}+\text{F1}^2}{\text{A1}}}}/\left(2\pi\sqrt{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right];$

(*Now need to collect coefficients for the triple integral*)

$\text{exponTrip}[\text{u\_},\text{v\_},\text{w\_}] = \frac{1}{2}\left(-w\left(\frac{w(-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.\right.$
$+\frac{v(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$
$\left.+\frac{u(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$
$-v\left(\frac{w(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.$
$+\frac{v(-\sigma\text{ac}^2+\sigma\text{aa}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$
$\left.+\frac{u(\sigma\text{ac}\sigma\text{bc}-\sigma\text{ab}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$
$-u\left(\frac{w(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.$

$+\frac{v(\sigma \mathrm{ac}\sigma \mathrm{bc}-\sigma \mathrm{ab}\sigma \mathrm{cc})}{-\sigma \mathrm{ac}^2\sigma \mathrm{bb}+2\sigma \mathrm{ab}\sigma \mathrm{ac}\sigma \mathrm{bc}-\sigma \mathrm{aa}\sigma \mathrm{bc}^2-\sigma \mathrm{ab}^2\sigma \mathrm{cc}+\sigma \mathrm{aa}\sigma \mathrm{bb}\sigma \mathrm{cc}}$

$+\frac{u(-\sigma \mathrm{bc}^2+\sigma \mathrm{bb}\sigma \mathrm{cc})}{-\sigma \mathrm{ac}^2\sigma \mathrm{bb}+2\sigma \mathrm{ab}\sigma \mathrm{ac}\sigma \mathrm{bc}-\sigma \mathrm{aa}\sigma \mathrm{bc}^2-\sigma \mathrm{ab}^2\sigma \mathrm{cc}+\sigma \mathrm{aa}\sigma \mathrm{bb}\sigma \mathrm{cc}}\Big)\Big)$

$+(-2\sigma m[t]z+J)u+\left(-z\sigma^2+H\right)u^2-2q\sigma v+(-2\sigma\ m[s]r)w+\left(-r\sigma^2\right)w^2;$

C00 = FullSimplify[exponTrip[0, 0, 0]];

Cu2 = −Coefficient $\left[\text{exponTrip}[u,0,0],u^2\right]$;

Cv2 = Coefficient $\left[\text{exponTrip}[0,v,0],v^2\right]$;

Cw2 = Coefficient $\left[\text{exponTrip}[0,0,w],w^2\right]$;

Cu = Coefficient[exponTrip[u, 0, 0], u];

Cv = Coefficient[exponTrip[0, v, 0], v];

Cw = Coefficient[exponTrip[0, 0, w], w];

Cuv = Coefficient[exponTrip[u, v, 0], uv];

Cuw = Coefficient[exponTrip[u, 0, w], uw];

Cvw = Coefficient[exponTrip[0, v, w], vw];

FullSimplify $\left[\text{exponTrip}[u,v,w]-\left(-\text{Cu2}*u^2+\text{Cv2}*v^2+\text{Cw2}*w^2+\text{Cu}*u+\text{Cv}*v\right.\right.$

$+\text{Cw}*w+\text{Cuv}*(u*v)+\text{Cuw}*(u*w)+\text{Cvw}*(v*w)+\text{C00})]$

0

$\kappa=\sqrt{\lambda^2+2q\sigma^2};$

J = JYt;

H = HYt2;

c1 = $-2zm[t]\sigma+J$; (*!!!!!!!!!!!!!!!!!!!!!*)

c2 = $-z\sigma^2+H$;

c3 = $-2q\sigma$;

gamma[z_, q_] = OC1 * OC2 * Exp[L] * Exp $\left[\frac{\text{B1(Cx1)}^2-\text{Dy1(A1Dy1+Cx1 F1)}}{4*(\text{A1B1})+\text{F1}^2}+\text{G1}\right]$;

(* Now to compute all the covariances*)

$\sigma 22=\frac{1}{2\kappa}\left(1-e^{-2\kappa T}\right);$

$\sigma 33=\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right);$

$\sigma 12=\int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+T)}\left(e^{2\kappa s}-1\right)\right)ds;$

$\sigma 13=\int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa t}-1\right)\right)ds;$

$\sigma 23=\frac{1}{2\kappa}\left(e^{\kappa(t-T)}-e^{-\kappa(t+T)}\right);$

$\sigma \mathrm{aa}=\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right);$

$\sigma \mathrm{ab}=\int_0^t m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s}-1\right)\right)ds;$

(*σ11 and σbb were precomputed*)

$\sigma 11 = -\frac{a^2\left(2+e^{-2t\kappa}-2e^{(t-T)\kappa}+e^{-2T\kappa}-2e^{-(t+T)\kappa}+2t\kappa-2T\kappa\right)}{2\kappa^3};$

$\sigma \text{bb} = \frac{a^2 e^{-2t\kappa}\left(-1+4e^{t\kappa}+e^{2t\kappa}(-3+2t\kappa)\right)}{2\kappa^3};$

(* We also need a few more covariances for second moment*)

$\sigma \text{cc} = \frac{1}{2\kappa}\left(1-e^{-2\kappa s}\right);$

$\sigma \text{ac} = \frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s}-1\right);$ (* $s \leq t$*)

$\sigma \text{bc} = -\frac{a e^{-(s+t)\kappa}\left(-1+e^{s\kappa}\right)\left(1+e^{s\kappa}-2e^{t\kappa}\right)}{2\kappa^2};$

Clear[S0, n, g, σBM, θ, v, μ];

ψ[n_]:= $\left(n * g + \frac{1}{2}n\text{^}2\sigma\text{BM^}2\right)$

cf[n_]:=Exp[$t * \psi[n]$]

(*Compute first moment*)

S0 = 110;

$\sigma = 5;$

$T = 1;$

$\lambda = 2;$

$b = 0;$

$a = 22;$

$\rho = 0;$

σBM = sigmaPrices; (*this is the diffusion coefficient in the vg stock price model*)

$\mu = 0.1;$

$g = \mu - 0.5 * \sigma$BM^2;

μx1 = 0;

μx2 = 0;

μx3 = 0;

μa = 0;

μb = 0;

Phi[z_, q_] = Exp$\left[-z(m[t])^2 - q\int_0^T (m[s])^2\, ds + \frac{(\lambda-\kappa)T}{2} - z * b - q * b * T\right]$ * gamma[z, q];

dzPhi[q_, t_] = Derivative[1, 0][Phi][z, q]/.$z \to 0$;

moment1 = NIntegrate[−S0 * cf[1]Exp[0.5 * ρ^2 * σBM^2 * t]dzPhi[q, t], {q, 0, ∞}, {t, 0, T}]

(*Nowtosetupthesecondmoment

*)

$$\text{OC2trip} = \frac{1}{\sqrt{\text{Cu2}}\sqrt{-\frac{\text{Cuv}^2+4\text{Cu2Cv2}}{\text{Cu2}}}}2\pi/$$

$$\left(\sqrt{\frac{-\text{Cuw}^2\text{Cv2}+\text{CuvCuwCvw}+\text{Cu2Cvw}^2-\text{Cuv}^2\text{Cw2}-4\text{Cu2Cv2Cw2}}{\text{Cuv}^2\pi+4\text{Cu2Cv2}\pi}}\right)\Big/$$

$$\left(2\sqrt{2}\pi^{3/2}\sqrt{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right);$$

$$\text{gamma2}[\text{z\_},\text{r\_},\text{q\_}] = \text{OC1} * \text{OC2trip} * \text{Exp}[L] * \text{Exp}\Big[-\left(\text{Cuw}^2\left(\text{Cv}^2-4\text{C00Cv2}\right)+4\text{C00Cu2Cvw}^2-\right.$$

$$4\text{Cu2CvCvwCw}+\text{Cuv}^2\text{Cw}^2+4\text{Cu2Cv2Cw}^2+\text{Cuw}(4\text{C00CuvCvw}-2\text{CuCvCvw}-2\text{CuvCvCw}+4\text{CuCv2Cw})$$

$$-4\text{C00Cuv}^2\text{Cw2}+4\text{Cu2Cv}^2\text{Cw2}$$

$$\left.-16\text{C00Cu2Cv2Cw2}+\text{CuCuv}(-2\text{CvwCw}+4\text{CvCw2})+\text{Cu}^2\left(\text{Cvw}^2-4\text{Cv2Cw2}\right)\right)\Big/$$

$$\left(4\left(\text{Cuw}^2\text{Cv2}-\text{CuvCuwCvw}-\text{Cu2Cvw}^2+\text{Cuv}^2\text{Cw2}+4\text{Cu2Cv2Cw2}\right)\right)\Big];$$

$$\text{Phi2}[\text{z\_},\text{r\_},\text{q\_}] = \text{Exp}\left[-z(m[t])^2-r(m[s])^2-q\int_0^T(m[s])^2\,ds+\frac{(\lambda-\kappa)T}{2}\right] * \text{gamma2}[z,r,q];$$

$$\text{dzdrPhi2}[\text{q\_},\text{t\_},\text{s\_}] = ((\text{Derivative}[1,1,0][\text{Phi2}][z,r,q]/.z\to 0)/.r\to 0);$$

$$\text{moment2} = \text{NIntegrate}\big[\text{S0}^2q\text{Exp}[s*\psi[2]+(t-s)*\psi[1]$$

$$+0.5*\rho\text{\^{}}2*\sigma\text{BM\^{}}2*(s+t)]\text{dzdrPhi2}[q,t,s],\{q,0,\infty\},\{t,0,T\},\{s,0,t\}]$$

$$+\text{NIntegrate}\big[\text{S0}^2q\text{Exp}[t*\psi[2]+(s-t)*\psi[1]$$

$$+0.5*\rho\text{\^{}}2*\sigma\text{BM\^{}}2*(s+t)]\text{dzdrPhi2}[q,s,t],\{q,0,\infty\},\{t,0,T\},\{s,t,T\}];$$

TimeUsed[]

Print["First moment = " <> ToString[moment1]]

Print["Second moment = " <> ToString[moment2]]

Print["Variance = " <> ToString[moment2 – moment1^2]]

Print["SD = " <> ToString[Sqrt[moment2 – moment1^2]]]

Print["mu tilde = " <> ToString[Log[moment1/S0]/T]]

$$\text{Print}\Big[\text{“sig tilde = ”} <> \text{ToString}\Big[\text{Sqrt}\Big[\text{Log}\Big[\frac{(\text{moment2}-\text{moment1\^{}2})+(\text{moment1})^2}{(\text{moment1})^2}\Big]\Big/T\Big]\Big]\Big]$$

Analytical approximation in response to the change of correlation

C.3 Analytical approximation under the Geometric Lévy model, $\rho = 0.3$

sigmaPrices = {0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1,
0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21, 0.22,
0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31, 0.32, 0.33, 0.34,
0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46,
0.47, 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.56, 0.57, 0.58,
0.59, 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69, 0.7,
0.71, 0.72, 0.73, 0.74, 0.75, 0.76, 0.77, 0.78, 0.79, 0.8, 0.81, 0.82,
0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89, 0.9, 0.91, 0.92, 0.93, 0.94,
0.95, 0.96, 0.97, 0.98, 0.99, 1};

$m[\mathrm{t\_}] = ae^{-\lambda t} + a\left(1 - e^{-\lambda t}\right);$

Needs[“MultivariateStatistics`”]

$$\mathrm{PDF}\left[\mathrm{MultinormalDistribution}\left[\left\{\mu\mathrm{x1} + \frac{\sigma 13}{\sigma 33}(z - \mu\mathrm{x3}), \mu\mathrm{x2} + \frac{\sigma 23}{\sigma 33}(z - \mu\mathrm{x3})\right\}\right.\right.$$
$$\left.\left., \left\{\left\{\sigma 11 - \frac{(\sigma 13)^2}{\sigma 33}, \sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}\right\}, \left\{\sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}, \sigma 22 - \frac{(\sigma 23)^2}{\sigma 33}\right\}\right\}\right], \{x, y\}\right];$$

$$\mathrm{expon}[\mathrm{x\_}, \mathrm{y\_}] = \frac{1}{2}\left(-\left(\frac{\left(y - \mu\mathrm{x2} - \frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\left(-\sigma 12 + \frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.\right.$$
$$\left.+ \frac{\left(x - \mu\mathrm{x1} - \frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(\sigma 22 - \frac{\sigma 23^2}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(x - \mu\mathrm{x1} - \frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)$$
$$- \left(\frac{\left(\sigma 11 - \frac{\sigma 13^2}{\sigma 33}\right)\left(y - \mu\mathrm{x2} - \frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right.$$
$$\left.\left.+ \frac{\left(x - \mu\mathrm{x1} - \frac{(z-\mu\mathrm{x3})\sigma 13}{\sigma 33}\right)\left(-\sigma 12 + \frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\left(y - \mu\mathrm{x2} - \frac{(z-\mu\mathrm{x3})\sigma 23}{\sigma 33}\right)\right)$$
$$-2q\sigma x + \frac{(\kappa - \lambda)y^2}{2};$$

$G = \mathrm{FullSimplify}[\mathrm{expon}[0, 0]];$

$F = \mathrm{FullSimplify}[\mathrm{Coefficient}[\mathrm{expon}[x, y], xy]];$

$A = -\mathrm{FullSimplify}\left[\mathrm{Coefficient}\left[\mathrm{expon}[x, 0], x^2\right]\right];$

$$B = \text{FullSimplify}\left[\text{Coefficient}\left[\text{expon}[0,y],y^2\right]\right];$$

$$\text{Cx} = \text{FullSimplify}[\text{Coefficient}[\text{expon}[x,0],x]];$$

$$\text{Dy} = \text{FullSimplify}[\text{Coefficient}[\text{expon}[0,y],y]];$$

$$\text{JYt} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G,z\right]\right];$$

$$\text{HYt2} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G,z^2\right]\right];$$

$$\text{constZ}[\text{z\_}] = \frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G;$$

$$L = \text{FullSimplify}[\text{constZ}[0]];$$

$$\text{FullSimplify}\left[\frac{B(\text{Cx})^2-\text{Dy}(A\text{Dy}+\text{Cx}F)}{4*(AB)+F^2}+G-\text{HYt2}*z^2-\text{JYt}*z-L\right]$$

0

$$\text{FullSimplify}\left[\text{expon}[x,y]-\left((\text{Cx}*x)+(\text{Dy}*y)+(F*(xy))+\left(-A*x^2\right)+\left(B*y^2\right)+G\right)\right]$$

0

$$\text{OC1} = \text{FullSimplify}\left[\frac{2\pi}{\sqrt{A}\sqrt{-\frac{4AB+F^2}{A}}}/\left(2\pi\sqrt{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\right];$$

$$\text{PDF}[\text{MultinormalDistribution}[\{\mu\text{a},\mu\text{b}\},\{\{\sigma\text{aa},\sigma\text{ab}\},\{\sigma\text{ab},\sigma\text{bb}\}\}],\{x,y\}];$$

$$\text{expon1}[\text{x\_},\text{y\_}] = \frac{1}{2}\left(-(y-\mu\text{b})\left(\frac{(y-\mu\text{b})\sigma\text{aa}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}-\frac{(x-\mu\text{a})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)-(x-\mu\text{a})\left(-\frac{(y-\mu\text{b})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}+\frac{(x-\mu\text{a})\sigma\text{bb}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right)$$
$$+(\text{c1})x+(\text{c2})x^2+(\text{c3})y;$$

$$\text{G1} = \text{FullSimplify}[\text{expon1}[0,0]];$$

$$\text{F1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,y],xy]];$$

$$\text{A1} = -\text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[x,0],x^2\right]\right];$$

$$\text{B1} = \text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[0,y],y^2\right]\right];$$

$$\text{Cx1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,0],x]];$$

$$\text{Dy1} = \text{FullSimplify}[\text{Coefficient}[\text{expon1}[0,y],y]];$$

$$\text{FullSimplify}\left[\text{expon1}[x,y]-\left((\text{Cx1}*x)+(\text{Dy1}*y)+(\text{F1}*(xy))+\left(-\text{A1}*x^2\right)+\left(\text{B1}*y^2\right)+\text{G1}\right)\right]$$

0

$$\text{OC2} = \text{FullSimplify}\left[\frac{2\pi}{\sqrt{\text{A1}}\sqrt{-\frac{4\text{A1B1}+\text{F1}^2}{\text{A1}}}}/\left(2\pi\sqrt{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right];$$

$$\text{exponTrip}[\text{u\_},\text{v\_},\text{w\_}] = \frac{1}{2}\left(-w\left(\frac{w\left(-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}\right)}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.\right.$$
$$+\frac{v(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$$
$$\left.+\frac{u(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$$

$$-v\left(\frac{w(\sigma\mathrm{ab}\sigma\mathrm{ac}-\sigma\mathrm{aa}\sigma\mathrm{bc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\right.$$
$$+\frac{v\left(-\sigma\mathrm{ac}^2+\sigma\mathrm{aa}\sigma\mathrm{cc}\right)}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$
$$\left.+\frac{u(\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{ab}\sigma\mathrm{cc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\right)$$
$$-u\left(\frac{w(-\sigma\mathrm{ac}\sigma\mathrm{bb}+\sigma\mathrm{ab}\sigma\mathrm{bc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\right.$$
$$+\frac{v(\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{ab}\sigma\mathrm{cc})}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}$$
$$\left.\left.+\frac{u\left(-\sigma\mathrm{bc}^2+\sigma\mathrm{bb}\sigma\mathrm{cc}\right)}{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\right)\right)$$
$$+(-2\sigma m[t]z+J)u+\left(-z\sigma^2+H\right)u^2-2q\sigma v+(-2\sigma\ m[s]r)w+\left(-r\sigma^2\right)w^2;$$

$\mathrm{C00}=\mathrm{FullSimplify}[\mathrm{exponTrip}[0,0,0]];$

$\mathrm{Cu2}=-\mathrm{Coefficient}\left[\mathrm{exponTrip}[u,0,0],u^2\right];$

$\mathrm{Cv2}=\mathrm{Coefficient}\left[\mathrm{exponTrip}[0,v,0],v^2\right];$

$\mathrm{Cw2}=\mathrm{Coefficient}\left[\mathrm{exponTrip}[0,0,w],w^2\right];$

$\mathrm{Cu}=\mathrm{Coefficient}[\mathrm{exponTrip}[u,0,0],u];$

$\mathrm{Cv}=\mathrm{Coefficient}[\mathrm{exponTrip}[0,v,0],v];$

$\mathrm{Cw}=\mathrm{Coefficient}[\mathrm{exponTrip}[0,0,w],w];$

$\mathrm{Cuv}=\mathrm{Coefficient}[\mathrm{exponTrip}[u,v,0],uv];$

$\mathrm{Cuw}=\mathrm{Coefficient}[\mathrm{exponTrip}[u,0,w],uw];$

$\mathrm{Cvw}=\mathrm{Coefficient}[\mathrm{exponTrip}[0,v,w],vw];$

$\mathrm{FullSimplify}\left[\mathrm{exponTrip}[u,v,w]-\left(-\mathrm{Cu2}*u^2+\mathrm{Cv2}*v^2+\mathrm{Cw2}*w^2+\mathrm{Cu}*u+\mathrm{Cv}*v\right.\right.$

$\left.\left.+\mathrm{Cw}*w+\mathrm{Cuv}*(u*v)+\mathrm{Cuw}*(u*w)+\mathrm{Cvw}*(v*w)+\mathrm{C00}\right)\right]$

0

$\kappa=\sqrt{\lambda^2+2q\sigma^2};$

$J=\mathrm{JYt};$

$H=\mathrm{HYt2};$

$\mathrm{c1}=-2zm[t]\sigma+J;$

$\mathrm{c2}=-z\sigma^2+H;$

$\mathrm{c3}=-2q\sigma;$

$\mathrm{gamma}[\mathrm{z\_},\mathrm{q\_}]=\mathrm{OC1}*\mathrm{OC2}*\mathrm{Exp}[L]*\mathrm{Exp}\left[\frac{\mathrm{B1}(\mathrm{Cx1})^2-\mathrm{Dy1}(\mathrm{A1Dy1}+\mathrm{Cx1\ F1})}{4*(\mathrm{A1B1})+\mathrm{F1}^2}+\mathrm{G1}\right];$

$\sigma 22=\frac{1}{2\kappa}\left(1-e^{-2\kappa T}\right);$

$\sigma 33=\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right);$

$\sigma 12=\int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+T)}\left(e^{2\kappa s}-1\right)\right)ds;$

$\sigma 13=\int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa t}-1\right)\right)ds;$

$\sigma 23 = \frac{1}{2\kappa}\left(e^{\kappa(t-T)} - e^{-\kappa(t+T)}\right);$

$\sigma \mathrm{aa} = \frac{1}{2\kappa}\left(1 - e^{-2\kappa t}\right);$

$\sigma \mathrm{ab} = \int_0^t m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s} - 1\right)\right)ds;$

$\sigma 11 = -\frac{a^2\left(2+e^{-2t\kappa}-2e^{(t-T)\kappa}+e^{-2T\kappa}-2e^{-(t+T)\kappa}+2t\kappa-2T\kappa\right)}{2\kappa^3};$

$\sigma \mathrm{bb} = \frac{a^2e^{-2t\kappa}\left(-1+4e^{t\kappa}+e^{2t\kappa}(-3+2t\kappa)\right)}{2\kappa^3};$

$\sigma \mathrm{cc} = \frac{1}{2\kappa}\left(1 - e^{-2\kappa s}\right);$

$\sigma \mathrm{ac} = \frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s} - 1\right);$ (* s ≤ t*)

$\sigma \mathrm{bc} = -\frac{ae^{-(s+t)\kappa}\left(-1+e^{s\kappa}\right)\left(1+e^{s\kappa}-2e^{t\kappa}\right)}{2\kappa^2};$

Clear[S0, n, g, σVG, σBM, θ, v, μ];

ψ[n_]:=$\left(n * g + \frac{1}{2}n\text{^}2\sigma\mathrm{RHO}\text{^}2 - \mathrm{Log}\left[1 - n\theta v + \frac{(n\sigma \mathrm{VG})\text{^}2v}{2}\right]/v\right)$

cf[n_]:=Exp[$t * \psi[n]$]

S0 = 110;

σ = 5;

T = 1;

λ = 2;

b = 0;

a = 22;

ρ = 0.3;

σVG = 0.1; (*control parameter for skewness*)

σBM = sigmaPrices; (*this is the diffusion coefficient in the vg stock price model*)

σRHO = σBM * $\left(\sqrt{1 - \rho\text{^}2}\right)$;

v = 0.1; (*control parameter for kurtosis*)

θ = −0.14; (*measure of symmetry*)

μ = 0.1;

$g = \mu - 0.5 * \sigma\mathrm{BM}\text{^}2 + \mathrm{Log}\left[1 - \theta v + \frac{(\sigma \mathrm{VG})\text{^}2v}{2}\right]/v;$

μx1 = 0;

μx2 = 0;

μx3 = 0;

μa = 0;

μb = 0;

(*ψ[1]; *)

Phi[z_, q_] = Exp$\left[-z(m[t])^2 - q\int_0^T (m[s])^2\,ds + \frac{(\lambda-\kappa)T}{2} - z*b - q*b*T\right]$ * gamma[z, q];

dzPhi[q_, t_] = Derivative[1, 0][Phi][z, q]/.$z \to 0$;

Timing[moment1 = NIntegrate[−S0 * cf[1]Exp[0.5 * ρ^2 * σBM^2 * t]dzPhi[q, t], {q, 0, ∞}, {t, 0, T}]]

(*Nowtosetupthesecondmoment

*)

$$\mathrm{OC2trip} = \frac{1}{\sqrt{\mathrm{Cu2}}\sqrt{-\frac{\mathrm{Cuv}^2+4\mathrm{Cu2Cv2}}{\mathrm{Cu2}}}}2\pi/$$
$$\left(\sqrt{\frac{-\mathrm{Cuw}^2\mathrm{Cv2}+\mathrm{CuvCuwCvw}+\mathrm{Cu2Cvw}^2-\mathrm{Cuv}^2\mathrm{Cw2}-4\mathrm{Cu2Cv2Cw2}}{\mathrm{Cuv}^2\pi+4\mathrm{Cu2Cv2}\pi}}\right)\Big/$$
$$\left(2\sqrt{2}\pi^{3/2}\sqrt{-\sigma\mathrm{ac}^2\sigma\mathrm{bb}+2\sigma\mathrm{ab}\sigma\mathrm{ac}\sigma\mathrm{bc}-\sigma\mathrm{aa}\sigma\mathrm{bc}^2-\sigma\mathrm{ab}^2\sigma\mathrm{cc}+\sigma\mathrm{aa}\sigma\mathrm{bb}\sigma\mathrm{cc}}\right);$$

gamma2[z_, r_, q_] = OC1 * OC2trip * Exp[L] * Exp$\Big[-\big(\mathrm{Cuw}^2\left(\mathrm{Cv}^2 - 4\mathrm{C00Cv2}\right) + 4\mathrm{C00Cu2Cvw}^2 -$

$4\mathrm{Cu2CvCvwCw} + \mathrm{Cuv}^2\mathrm{Cw}^2 + 4\mathrm{Cu2Cv2Cw}^2 + \mathrm{Cuw}(4\mathrm{C00CuvCvw} - 2\mathrm{CuCvCvw} - 2\mathrm{CuvCvCw} + 4\mathrm{CuCv2Cw})$

$-4\mathrm{C00Cuv}^2\mathrm{Cw2} + 4\mathrm{Cu2Cv}^2\mathrm{Cw2}$

$-16\mathrm{C00Cu2Cv2Cw2} + \mathrm{CuCuv}(-2\mathrm{CvwCw} + 4\mathrm{CvCw2}) + \mathrm{Cu}^2\left(\mathrm{Cvw}^2 - 4\mathrm{Cv2Cw2}\right)\big)/$

$\left(4\left(\mathrm{Cuw}^2\mathrm{Cv2} - \mathrm{CuvCuwCvw} - \mathrm{Cu2Cvw}^2 + \mathrm{Cuv}^2\mathrm{Cw2} + 4\mathrm{Cu2Cv2Cw2}\right)\right)\Big]$;

Phi2[z_, r_, q_] = Exp$\left[-z(m[t])^2 - r(m[s])^2 - q\int_0^T (m[s])^2\,ds + \frac{(\lambda-\kappa)T}{2}\right]$ * gamma2[z, r, q];

dzdrPhi2[q_, t_, s_] = ((Derivative[1, 1, 0][Phi2][z, r, q]/.$z \to 0$)/.$r \to 0$);

moment2 =

NIntegrate$\left[\mathrm{S0}^2 q\right.$Exp[$s$ * ψ[2] + ($t - s$) * ψ[1] + 0.5 * ρ^2 * σBM^2 * ($s + t$)]dzdrPhi2[q, t, s],

{q, 0, ∞}, {t, 0, T}, {s, 0, t}]+

NIntegrate$\left[\mathrm{S0}^2 q\right.$Exp[$t$ * ψ[2] + ($s - t$) * ψ[1] + 0.5 * ρ^2 * σBM^2 * ($s + t$)]dzdrPhi2[q, s, t],

{q, 0, ∞}, {t, 0, T}, {s, t, T}];

Print["First moment = " <> ToString[moment1]]

Print["Second moment = " <> ToString[moment2]]

Print["Variance = " <> ToString[moment2 − moment1^2]]

Print["SD = " <> ToString[Sqrt[moment2 − moment1^2]]]

Print["mu tilde = " <> ToString[Log[moment1/S0]/T]]

Print$\left[\right.$"sig tilde = " <> ToString$\left[\mathrm{Sqrt}\left[\mathrm{Log}\left[\frac{(\mathrm{moment2-moment1\string^2})+(\mathrm{moment1})^2}{(\mathrm{moment1})^2}\right]\Big/T\right]\right]\left.\right]$

C.4 Analytical approximation under the Geometric Lévy model, $\rho = 0.5$

sigmaPrices = {0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1,
0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21, 0.22,
0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31, 0.32, 0.33, 0.34,
0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46,
0.47, 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.56, 0.57, 0.58,
0.59, 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69, 0.7,
0.71, 0.72, 0.73, 0.74, 0.75, 0.76, 0.77, 0.78, 0.79, 0.8, 0.81, 0.82,
0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89, 0.9, 0.91, 0.92, 0.93, 0.94,
0.95, 0.96, 0.97, 0.98, 0.99, 1}

$m[\text{t\_}] = ae^{-\lambda t} + a\left(1 - e^{-\lambda t}\right);$

Needs[“MultivariateStatistics”]

$$\text{PDF}\left[\text{MultinormalDistribution}\left[\left\{\mu\text{x1} + \frac{\sigma 13}{\sigma 33}(z - \mu\text{x3}), \mu\text{x2} + \frac{\sigma 23}{\sigma 33}(z - \mu\text{x3})\right\}, \left\{\left\{\sigma 11 - \frac{(\sigma 13)^2}{\sigma 33}, \sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}\right\}, \left\{\sigma 12 - \frac{\sigma 13\sigma 23}{\sigma 33}, \sigma 22 - \frac{(\sigma 23)^2}{\sigma 33}\right\}\right\}\right], \{x, y\}\right];$$

$$\begin{aligned}\text{expon}[\text{x\_}, \text{y\_}] = \frac{1}{2}\Bigg(&-\Bigg(\frac{\left(y - \mu\text{x2} - \frac{(z-\mu\text{x3})\sigma 23}{\sigma 33}\right)\left(-\sigma 12 + \frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}} \\ &+ \frac{\left(x - \mu\text{x1} - \frac{(z-\mu\text{x3})\sigma 13}{\sigma 33}\right)\left(\sigma 22 - \frac{\sigma 23^2}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\Bigg)\left(x - \mu\text{x1} - \frac{(z-\mu\text{x3})\sigma 13}{\sigma 33}\right) \\ &-\Bigg(\frac{\left(\sigma 11 - \frac{\sigma 13^2}{\sigma 33}\right)\left(y - \mu\text{x2} - \frac{(z-\mu\text{x3})\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}} \\ &+ \frac{\left(x - \mu\text{x1} - \frac{(z-\mu\text{x3})\sigma 13}{\sigma 33}\right)\left(-\sigma 12 + \frac{\sigma 13\sigma 23}{\sigma 33}\right)}{-\sigma 12^2 + \sigma 11\sigma 22 - \frac{\sigma 13^2\sigma 22}{\sigma 33} + \frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33} - \frac{\sigma 11\sigma 23^2}{\sigma 33}}\Bigg)\left(y - \mu\text{x2} - \frac{(z-\mu\text{x3})\sigma 23}{\sigma 33}\right)\Bigg) \\ &-2q\sigma x + \frac{(\kappa - \lambda)y^2}{2}; (\text{*!!!!!!!!!!!!!!!!!!!!!!!!!!*})\end{aligned}$$

G = FullSimplify[expon[0, 0]];

F = FullSimplify[Coefficient[expon[x, y], xy]];

A = −FullSimplify [Coefficient [expon[x, 0], x^2]] ;

B = FullSimplify [Coefficient [expon[0, y], y^2]] ;

Cx = FullSimplify[Coefficient[expon[x, 0], x]];

Dy = FullSimplify[Coefficient[expon[0, y], y]];

$$\text{JYt} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2 - \text{Dy}(A\text{Dy} + \text{Cx}F)}{4*(AB) + F^2} + G, z\right]\right];$$

$$\text{HYt2} = \text{FullSimplify}\left[\text{Coefficient}\left[\frac{B(\text{Cx})^2 - \text{Dy}(A\text{Dy} + \text{Cx}F)}{4*(AB) + F^2} + G, z^2\right]\right];$$

$$\text{constZ}[\text{z\_}] = \frac{B(\text{Cx})^2 - \text{Dy}(A\text{Dy} + \text{Cx}F)}{4*(AB) + F^2} + G;$$

L = FullSimplify[constZ[0]];

(*Checkthatthecoefficients$H$, $J$and$L$arecorrect*)

$$\text{FullSimplify}\left[\frac{B(\text{Cx})^2 - \text{Dy}(A\text{Dy} + \text{Cx}F)}{4*(AB) + F^2} + G - \text{HYt2} * z^2 - \text{JYt} * z - L\right]$$

0

(*CheckthatthecoefficientsA, B, C, D, FandGarecorrect*)

$\text{FullSimplify}\left[\text{expon}[x,y]-\left((\text{Cx}*x)+(\text{Dy}*y)+(F*(xy))+\left(-A*x^2\right)+\left(B*y^2\right)+G\right)\right]$

0

(*constantC_*inlatexdocument*)

$\text{OC1}=\text{FullSimplify}\left[\frac{2\pi}{\sqrt{A}\sqrt{-\frac{4AB+F^2}{A}}}/\left(2\pi\sqrt{-\sigma 12^2+\sigma 11\sigma 22-\frac{\sigma 13^2\sigma 22}{\sigma 33}+\frac{2\sigma 12\sigma 13\sigma 23}{\sigma 33}-\frac{\sigma 11\sigma 23^2}{\sigma 33}}\right)\right];$

(*——————————–

Nowtheouterintegral

——————————

*)

$\text{PDF}[\text{MultinormalDistribution}[\{\mu\text{a},\mu\text{b}\},\{\{\sigma\text{aa},\sigma\text{ab}\},\{\sigma\text{ab},\sigma\text{bb}\}\}],\{x,y\}];$

$\text{expon1}[\text{x\_},\text{y\_}]=\frac{1}{2}\left(-(y-\mu\text{b})\left(\frac{(y-\mu\text{b})\sigma\text{aa}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}-\frac{(x-\mu\text{a})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)-(x-\mu\text{a})\left(-\frac{(y-\mu\text{b})\sigma\text{ab}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}+\frac{(x-\mu\text{a})\sigma\text{bb}}{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right)$
$+(\text{c1})x+(\text{c2})x^2+(\text{c3})y;$

$\text{G1}=\text{FullSimplify}[\text{expon1}[0,0]];$

$\text{F1}=\text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,y],xy]];$

$\text{A1}=-\text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[x,0],x^2\right]\right];$

$\text{B1}=\text{FullSimplify}\left[\text{Coefficient}\left[\text{expon1}[0,y],y^2\right]\right];$

$\text{Cx1}=\text{FullSimplify}[\text{Coefficient}[\text{expon1}[x,0],x]];$

$\text{Dy1}=\text{FullSimplify}[\text{Coefficient}[\text{expon1}[0,y],y]];$

(*CheckthatthecoefficientsA1, B1, C1, D1, F1andG1arecorrect*)

$\text{FullSimplify}\left[\text{expon1}[x,y]-\left((\text{Cx1}*x)+(\text{Dy1}*y)+(\text{F1}*(xy))+\left(-\text{A1}*x^2\right)+\left(\text{B1}*y^2\right)+\text{G1}\right)\right]$

0

$\text{OC2}=\text{FullSimplify}\left[\frac{2\pi}{\sqrt{\text{A1}}\sqrt{-\frac{4\text{A1B1}+\text{F1}^2}{\text{A1}}}}/\left(2\pi\sqrt{-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}}\right)\right];$

(*Now need to collect coefficients for the triple integral*)

$\text{exponTrip}[\text{u\_},\text{v\_},\text{w\_}]=\frac{1}{2}\left(-w\left(\frac{w\left(-\sigma\text{ab}^2+\sigma\text{aa}\sigma\text{bb}\right)}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.\right.$
$+\frac{v(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$
$\left.+\frac{u(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$
$-v\left(\frac{w(\sigma\text{ab}\sigma\text{ac}-\sigma\text{aa}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.$
$+\frac{v\left(-\sigma\text{ac}^2+\sigma\text{aa}\sigma\text{cc}\right)}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$
$\left.+\frac{u(\sigma\text{ac}\sigma\text{bc}-\sigma\text{ab}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$
$-u\left(\frac{w(-\sigma\text{ac}\sigma\text{bb}+\sigma\text{ab}\sigma\text{bc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right.$
$+\frac{v(\sigma\text{ac}\sigma\text{bc}-\sigma\text{ab}\sigma\text{cc})}{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}$

$+\frac{u(-\sigma bc^2+\sigma bb\sigma cc)}{-\sigma ac^2\sigma bb+2\sigma ab\sigma ac\sigma bc-\sigma aa\sigma bc^2-\sigma ab^2\sigma cc+\sigma aa\sigma bb\sigma cc}\Big)\Big)$

$+(-2\sigma m[t]z+J)u+\left(-z\sigma^2+H\right)u^2-2q\sigma v+(-2\sigma\ m[s]r)w+\left(-r\sigma^2\right)w^2;$

C00 = FullSimplify[exponTrip[0, 0, 0]];

Cu2 = −Coefficient $\left[\text{exponTrip}[u,0,0],u^2\right]$;

Cv2 = Coefficient $\left[\text{exponTrip}[0,v,0],v^2\right]$;

Cw2 = Coefficient $\left[\text{exponTrip}[0,0,w],w^2\right]$;

Cu = Coefficient[exponTrip[u, 0, 0], u];

Cv = Coefficient[exponTrip[0, v, 0], v];

Cw = Coefficient[exponTrip[0, 0, w], w];

Cuv = Coefficient[exponTrip[u, v, 0], uv];

Cuw = Coefficient[exponTrip[u, 0, w], uw];

Cvw = Coefficient[exponTrip[0, v, w], vw];

FullSimplify $\left[\text{exponTrip}[u,v,w]-\left(-\text{Cu2}*u^2+\text{Cv2}*v^2+\text{Cw2}*w^2+\text{Cu}*u+\text{Cv}*v\right.\right.$

$\left.\left.+\text{Cw}*w+\text{Cuv}*(u*v)+\text{Cuw}*(u*w)+\text{Cvw}*(v*w)+\text{C00}\right)\right]$

0

$\kappa=\sqrt{\lambda^2+2q\sigma^2};$

J = JYt;

H = HYt2;

c1 = $-2zm[t]\sigma+J$; (*!!!!!!!!!!!!!!!!!!!!!*)

c2 = $-z\sigma^2+H$;

c3 = $-2q\sigma$;

gamma[z_, q_] = OC1 * OC2 * Exp[L] * Exp $\left[\frac{\text{B1}(\text{Cx1})^2-\text{Dy1}(\text{A1Dy1}+\text{Cx1 F1})}{4*(\text{A1B1})+\text{F1}^2}+\text{G1}\right]$;

(* Now to compute all the covariances*)

$\sigma 22=\frac{1}{2\kappa}\left(1-e^{-2\kappa T}\right);$

$\sigma 33=\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right);$

$\sigma 12=\int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+T)}\left(e^{2\kappa s}-1\right)\right)ds;$

$\sigma 13=\int_t^T m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa t}-1\right)\right)ds;$

$\sigma 23=\frac{1}{2\kappa}\left(e^{\kappa(t-T)}-e^{-\kappa(t+T)}\right);$

$\sigma aa=\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right);$

$\sigma ab=\int_0^t m[s]\left(\frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s}-1\right)\right)ds;$

(*σ11andσbbprecomputedincovariances.nb*)

$\sigma 11=-\frac{a^2\left(2+e^{-2t\kappa}-2e^{(t-T)\kappa}+e^{-2T\kappa}-2e^{-(t+T)\kappa}+2t\kappa-2T\kappa\right)}{2\kappa^3};$

$\sigma \text{bb} = \frac{a^2 e^{-2t\kappa}\left(-1+4e^{t\kappa}+e^{2t\kappa}(-3+2t\kappa)\right)}{2\kappa^3};$

(* We also need a few more covariances for second moment*)

$\sigma \text{cc} = \frac{1}{2\kappa}\left(1-e^{-2\kappa s}\right);$

$\sigma \text{ac} = \frac{1}{2\kappa}e^{-\kappa(s+t)}\left(e^{2\kappa s}-1\right);$ (* s ≤ t*)

$\sigma \text{bc} = -\frac{a e^{-(s+t)\kappa}(-1+e^{s\kappa})(1+e^{s\kappa}-2e^{t\kappa})}{2\kappa^2};$

Clear[S0, n, g, σVG, σBM, θ, v, μ];

ψ[n_]:= $\left(n * g + \frac{1}{2}n\text{^}2\sigma\text{RHO^}2 - \text{Log}\left[1 - n\theta v + \frac{(n\sigma\text{VG})\text{^}2v}{2}\right] / v\right)$

cf[n_]:=Exp[$t * \psi[n]$]

(*Compute first moment*)

S0 = 110;

σ = 5;

T = 1;

λ = 2;

b = 0;

a = 22;

ρ = 0.5;

σVG = 0.1; (*control parameter for skewness*)

σBM = sigmaPrices; (*this is the diffusion coefficient in the vg stock price model*)

σRHO = σBM * $\left(\sqrt{1-\rho\text{^}2}\right)$;

v = 0.1; (*control parameter for kurtosis*)

θ = −0.14; (*measure of symmetry*)

μ = 0.1;

$g = \mu - 0.5 * \sigma\text{BM^}2 + \text{Log}\left[1 - \theta v + \frac{(\sigma\text{VG})\text{^}2v}{2}\right] / v;$

μx1 = 0;

μx2 = 0;

μx3 = 0;

μa = 0;

μb = 0;

(*ψ[1];*)

Phi[z_, q_] = Exp$\left[-z(m[t])^2 - q\int_0^T (m[s])^2\, ds + \frac{(\lambda-\kappa)T}{2} - z * b - q * b * T\right]$ * gamma[z, q];

dzPhi[q_, t_] = Derivative[1, 0][Phi][z, q]/.$z \to 0$;

Timing[moment1 = NIntegrate[−S0 * cf[1]Exp[0.5 * ρ^2 * σBM^2 * t]dzPhi[q, t], {q, 0, ∞}, {t, 0, T}]]

(*Nowtosetupthesecondmoment

*)

$\text{OC2trip} = \frac{1}{\sqrt{\text{Cu2}}\sqrt{-\frac{\text{Cuv}^2+4\text{Cu2Cv2}}{\text{Cu2}}}}2\pi/$

$\left(\sqrt{\frac{-\text{Cuw}^2\text{Cv2}+\text{CuvCuwCvw}+\text{Cu2Cvw}^2-\text{Cuv}^2\text{Cw2}-4\text{Cu2Cv2Cw2}}{\text{Cuv}^2\pi+4\text{Cu2Cv2}\pi}}\right)\Big/$

$\left(2\sqrt{2}\pi^{3/2}\sqrt{-\sigma\text{ac}^2\sigma\text{bb}+2\sigma\text{ab}\sigma\text{ac}\sigma\text{bc}-\sigma\text{aa}\sigma\text{bc}^2-\sigma\text{ab}^2\sigma\text{cc}+\sigma\text{aa}\sigma\text{bb}\sigma\text{cc}}\right)$;

$\text{gamma2}[\text{z\_},\text{r\_},\text{q\_}] = \text{OC1} * \text{OC2trip} * \text{Exp}[L] * \text{Exp}\big[-\big(\text{Cuw}^2\left(\text{Cv}^2-4\text{C00Cv2}\right)+4\text{C00Cu2Cvw}^2$

$-4\text{Cu2CvCvwCw}+\text{Cuv}^2\text{Cw}^2$

$+4\text{Cu2Cv2Cw}^2+\text{Cuw}(4\text{C00CuvCvw}-2\text{CuCvCvw}-2\text{CuvCvCw}+4\text{CuCv2Cw})$

$-4\text{C00Cuv}^2\text{Cw2}+4\text{Cu2Cv}^2\text{Cw2}$

$-16\text{C00Cu2Cv2Cw2}+\text{CuCuv}(-2\text{CvwCw}+4\text{CvCw2})+\text{Cu}^2\left(\text{Cvw}^2-4\text{Cv2Cw2}\right)\big)\big/$

$\left(4\left(\text{Cuw}^2\text{Cv2}-\text{CuvCuwCvw}-\text{Cu2Cvw}^2+\text{Cuv}^2\text{Cw2}+4\text{Cu2Cv2Cw2}\right)\right)\big]$;

$\text{Phi2}[\text{z\_},\text{r\_},\text{q\_}] = \text{Exp}\left[-z(m[t])^2-r(m[s])^2-q\int_0^T(m[s])^2\,ds+\frac{(\lambda-\kappa)T}{2}\right] * \text{gamma2}[z,r,q]$;

$\text{dzdrPhi2}[\text{q\_},\text{t\_},\text{s\_}] = ((\text{Derivative}[1,1,0][\text{Phi2}][z,r,q]/.z\to 0)/.r\to 0)$;

moment2 =

$\text{NIntegrate}\left[\text{S0}^2q\text{Exp}[s*\psi[2]+(t-s)*\psi[1]+0.5*\rho\text{\^{}}2*\sigma\text{BM\^{}}2*(s+t)]\text{dzdrPhi2}[q,t,s],\right.$

$\{q,0,\infty\},\{t,0,T\},\{s,0,t\}]+$

$\text{NIntegrate}\left[\text{S0}^2q\text{Exp}[t*\psi[2]+(s-t)*\psi[1]+0.5*\rho\text{\^{}}2*\sigma\text{BM\^{}}2*(s+t)]\text{dzdrPhi2}[q,s,t],\right.$

$\{q,0,\infty\},\{t,0,T\},\{s,t,T\}]$;

Print[“First moment = ” <> ToString[moment1]]

Print[“Second moment = ” <> ToString[moment2]]

Print[“Variance = ” <> ToString[moment2 − moment1^2]]

Print[“SD = ” <> ToString[Sqrt[moment2 − moment1^2]]]

Print[“mu tilde = ” <> ToString[Log[moment1/S0]/T]]

Print[“sig tilde = ” <> ToString[Sqrt[Log[$\frac{(\text{moment2}-\text{moment1\^{}2})+(\text{moment1})^2}{(\text{moment1})^2}$]/$T$]]]

Appendix D

CD contents

The CD contains TWO folders. Folder 1 contains all Mathematica Codes for Analytical approximation into four subfolders.

Subfolder rho0 contains all Mathematica codes for the analytical approximation (moments, evolution of lognormal parameter by solving moments, option prices under both GIG and lognormal model) under the main model (geometric Lévy model) under the condition of $\rho = 0$ between the two Wiener processes. The exported analytical moments data from Mathematica notebook GL 2nd moment 100 rho0 are in the GL 2nd moment 100 rho0 CSV file. The exported option prices and moments values from MATLAB are in the VWAP GL MC rho0 CSV file.

Subfolder GBM contains all Mathematica codes for the analytical approximation (moments, option prices under both GIG and lognormal model) under the GBM model.The exported analytical moments data from Mathematica notebook GBM 2nd moment 100 rho0 are in the GBM 2nd moment 100 rho0 CSV file. The exported option prices and moments values from MATLAB are in the VWAP GBM MC rho0 CSV file.

Subfolder rho03 contains all Mathematica codes for the analytical approximation (moments, option prices under both GIG and lognormal model) under the main model (geometric Lévy model) under the condition of $\rho = 0.3$ between the two Wiener processes. The exported analytical moments data from Mathematica notebook GL 2nd moment 100 rho03 are in the GL 2nd moment 100 rho03 CSV file. The exported option prices and moments values from MATLAB are in the VWAP G MC rho03 file.

Subfolder rho05 contains all Mathematica codes for the analytical approximation (moments, option prices under both GIG and lognormal model) under the main model (geometric Lévy model) under the condition of $\rho = 0.5$ between the two Wiener processes. The exported analytical moments data from Mathematica notebook GL 2nd moment 100 rho05 are in the GL 2nd moment 100 rho05 CSV file. The exported option prices and moments values from MATLAB are in the VWAP GL MC rho05 file.

Folder 2 contains all MATLAB codes used in this thesis.

Bibliography

[1] Hyperbolic distribution. http://fedc.wiwi.hu-berlin.de/xplore/ebooks/html/csa/node236.html, December 2011.

[2] Albrecher, H., and Predota, M. On Asian option pricing for NIG Lévy processes. *J. Comput. Appl. Math. 172*, 1 (2004), 153–168.

[3] Avramidis, A. N., and L'Ecuyer, P. Efficient monte carlo and quasi–monte carlo option pricing under the variance gamma model. *Manage. Sci. 52* (December 2006), 1930–1944.

[4] Barndorff-Nielsen, O. Normal inverse gaussian distributions and the modelling of stock returns. Research Report 300, Department of Theorectical Statistics, Aarhus University, 1995.

[5] Barndorff-Nielsen, O., and Halgreen, C. Infinite divisibility of the hyperbolic and generalized inverse gaussian distributions. *Probability Theory and Related Fields 38* (1977), 309–311.

[6] Barndorff-Nielsen, O. E. Processes of normal inverse gaussian type. *Finance Stochastic Journal* (1998), 41–68.

[7] Bates, D. S. Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies 9*, 1 (1996), 69–107.

[8] Borovkov, K. *Elements of stochastic modelling.* World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[9] Brigo, D., and Mercurio, F. *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit (Springer Finance)*, 2nd ed. Springer, August 2006.

[10] Brigo, D., Mercurio, F., Rapisarda, F., and Scotti, R. Approximated moment-matching dynamics for basket-options pricing. *Quantitative Finance 4*, 1 (February 2004), 1–16.

[11] Buryak, A. Volume weighted average price options. Market Risk Quantitative Support group, Wholesale Banking, National Australia Bank, December 2011.

[12] Carr, P., Geman, H., Madan, D., and Yor, M. The fine structure of asset returns: An empirical investigation. *Journal of Business 75* (2002), 305–332.

[13] Carr, P., Geman, H., Madan, D., and Yor, M. Stochastic volatility for lévy processes. *Maths. Finance 13* (2003), pp.345–382.

[14] Chaubey, Y. P., Garrido, J., and Trudeau, S. On the computation of aggregate claims distributions: some new approximations. *Insurance Math. Econom. 23*, 3 (1998), 215–230.

[15] Cont, R., and Tankov, P. *Financial modelling with Jump processes.* Chapman and hall-CRC financial mathematics series, 2004.

[16] Crank, J., and Nicolson, P. A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type. In *Proceedings of the Cambridge Philosophical Society* (1947), vol. 43, pp. 50–64.

[17] Daniel, D. The distribution of a perpetuity with applications to risk theory and pension funding. *Scandinavian Actuarial Journal* (1990), 39–79.

[18] de Finetti, B. Sulle funzioni ad incremento aleatorio. *Rend. Acc. Naz. Lincei*, 2 (1929), 163–168.

[19] Delbaen, F., and Schachermayer, W. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen 312*, 2 (1998), 215–250.

[20] Di Nunno, G., Øksendal, B., and Proske, F. *Malliavin calculus for Lévy processes with applications to finance*. Universitext. Springer-Verlag, Berlin, 2009.

[21] Doléans-Dade, C. Quelques applications de la formulae de changement de variable pour les semi-martingales. *Probability Theory and Related Fields 16*, 3 (1970), 181–194.

[22] DUFRESNE, François, G.-H. U. S. E. S. Risk theory with the gamma process. *ASTIN Bulletin 21*, 2 (November 1991), 177–192.

[23] Eberlein, E. Applications of generalized hyperbolic lévy motion to finance. In *Lévy processes-Theory and Applications* (2001), S. I. R. O E. Barndorff-Nielsen, T. Mikosch, Ed., Birkhuser, pp. 319–336.

[24] Eberlein, E. *Jump-type Lévy processes. In Handbook of Financial Time Series.* Springer, 2007.

[25] Eberlein, E., and Keller, U. Hyperbolic distributions in finance. *BERNOULLI 1* (1995), 281–299.

[26] Embrechts, P. A property of the generalized inverse gaussian distribution with some applications. *Journal of Applied Probability 20* (1983), 537–544.

[27] Fama, E. F. Mandelbrot and the stable paretian hypothesis. *The Journal of Business 36* (1963), 420.

[28] Fletcher, C. A. J. *Finite Difference Schemes and Partial Differential Equations*, vol. 1. Spring Verlag, 1991.

[29] Föllmer, H., and Schweizer, M. Hedging of contingent claims under incomplete information. In *Applied stochastic analysis (London, 1989)*, vol. 5 of *Stochastics Monogr.* Gordon and Breach, 1991, pp. 389–414.

[30] Frittelli, M. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Math. Finance 10*, 1 (2000), 39–52.

[31] Fu, M. C. Variance-gamma and monte carlo. *Advances in Mathematical Finance* (2000), 21–34.

[32] Geman, H. Pure jump lévy processes for asset price modelling. *Journal of Banking and Finance 26* (2002), 1297–1316.

[33] Gerber, H. U., and Shiu, E. S. W. Option pricing by esscher transform. *Transactions of Society of Actuaries 46* (1994).

[34] Glasserman, P. *Monte Carlo Methods in Financial Engineering*, vol. 53. Springer, 2004.

[35] Goll, T., and Kallsen, J. Optimal portfolios for logarithmic utility. In *STOCHASTIC PROCESSES AND THEIR APPLICATIONS 89* (1999), pp. 31–48.

[36] Ibragimov, I. A., and Khasminskii, R. Z. *Statistical estimation–asymptotic theory.* Springer Verlag, 1981.

[37] Jørgensen, B. *Statistical properties of the generalized inverse Gaussian distribution*, vol. 9 of *Lecture Notes in Statistics*. Springer Verlag, New York, 1982.

[38] Kaishev, V. K., and Dimitrova, D. S. Dirichlet bridge sampling for the variance gamma process: Pricing path-dependent options. *Manage. Sci. 55* (March 2009), 483–496.

[39] Karatzas, I., and Shreve, S. E. *Brownian motion and stochastic calculus*, vol. 113 of *Graduate Texts in Mathematics*. Springer Verlag, New York, 1991.

[40] Klein, I. A fundamental theorem of asset pricing for large financial markets. *Math. Finance 10*, 4 (2000), 443–458.

[41] KOPONEN, I. Analytic approach to the problem of convergence of truncated lévy flights towards the gaussian stochastic process. *Physical Review E 52* (1995), 1197–1199.

[42] Kou, S. A jump-diffusion model for option pricing. *Management Science 48* (2002), 1086–1101.

[43] Kyprianou, A. E., and Schoutens, W. *Exotic option pricing and advanced Lévy models*. John Wiley & Sons Ltd., 3005.

[44] Lipster, R., and Shiryaev, A. *Statistics of Random Processes*. Springer Verlag, Berlin, 2001.

[45] Liptser, R., and Shiryayev, A. *Theory of martingales*, vol. 49 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by K. Dzjaparidze.

[46] Lo, K.-H., Wang, K., and Hsu, M.-F. Pricing european asian options with skewness and kurtosis in the underlying distribution. *Journal of Futures Markets 28*, 6 (2008), 598–616.

[47] Madan, D., Carr, P., and Chang, E. The variance gamma process and option pricing. *European FinanceReview 2* (1998), 79–105.

[48] Madan, D. B., and Seneta, E. The variance gamma (V.G.) model for share market return. *Journal of Business 63* (1990), 511–524.

[49] Mandelbrot, B. The variation of certain speculative prices. *The Journal of Business 36*, 4 (1963), 394.

[50] Merton, R. Option pricing when underlying stock returns are discountinuous. *Journal of Financial Economics 3*, 125–144.

[51] Milevsky, M. A., and Posner, S. E. Asian options, the sum of lognormals, and the reciprocal gamma distribution. *Journal of financial and quantitative analysis 33*, 3 (1998), 409–422.

[52] Miline, F., and Madan, D. Option pricing with V.G. martingale component. Tech. rep.

[53] Mitchell, A., and Griffiths, D. *The Finite Difference Method in Partial Differential Equations*. John Wiley and Sons, 1980.

[54] Miyahara, Y. Canonical martingale measures of incomplete assets markets. In *Probability theory and mathematical statistics (Tokyo,1995)*. World Sci. Publ., River Edge, NJ, 1996, pp. 343–352.

[55] Miyahara, Y. Martingale measures for the geometric lévy process models. *Discussion papers in Economics, Nagoya City University*, 431 (2005), 1–14.

[56] Miyahara, Y., and Novikov, A. Geometric Lévy process pricing model. *Tr. Mat. Inst. Steklova 237*, Stokhast. Finans. Mat. (2002), 185–200.

[57] Mood, A. M., Graybill, F. A., and Boes, D. C. *Introduction to the Theory of Statistics*. McGraw-Hill, 1974.

[58] Morales, M. *Generalized risk processes and Lévy modelling in risk theory*. PhD thesis, Concordia University, 2003.

[59] Morton, K. W., and Msyers, D. F. *Numerical Solution of Partial Differential Equations An Introduction*, 2nd ed. Cambridge University Press, 2005.

[60] Musiela, M., and Rutkowski, M. *Martingale Methods in Financial Modelling*. Springer Verlag, New York, 2005.

[61] Novikov, A. A. Estimation of the parameters of diffusion processes. *Studia SciMath.Hungar 7* (1972), 201–209.

[62] Novikov, A. A., Ling, T. G., and Kordzakhia, N. Pricing of volume-weighted average options: analytical approximation and numerical results. 2010.

[63] Øksendal, B. *Stochastic differential equations: an introduction with applications*, 5th ed. Universitext. Springer Verlag, Berlin, 2000.

[64] Papapantoleon, A. An introduction to lévy processes with applications in finance, Apr. 2008.

[65] Peaceman, D. W., and Rachford, H. The Numerical Solution of Parabolic and Elliptic Differential Equations. *Journal of the Society for Industrial and Applied Mathematics 3*, 1 (1955).

[66] Platen, E., and Bruti-Liberati, N. *Numerical solution of stochastic differential equations with jumps in finance*. Stochastic Modelling and Applied Probability. Springer Verlag, Berlin, 2010.

[67] Platen, E., and Heath, D. *A benchmark approach to quantitative finance*. Springer Finance. Springer-Verlag, 2006.

[68] Praetz, P. D. The distribution of share price changes. *The Journal of Business 45*, 1 (January 1972), 49–55.

[69] Press, S. J. A compound events model for security prices. *The Journal of Business 40* (1967), 317.

[70] Protter, P. *Stochastic Integration and Differential Equations*, second ed. Springer Verlag, 2005.

[71] Revuz, D., and Yor, M. *Continuous Martingales and Brownian Motion*. Springer Verlag, Berlin.

[72] Ribeiro, C., and Webber, N. Path-dependent options: Extending the monte carlo simulation approach. *Journal of Computational Finance 43*, 11 (2004), 81–100.

[73] Samuelson, P. A. Rational theory of warrant pricing. *Industrial Management Review 6* (1965), 13–31.

[74] Sato, K. *Lévy processes and infinitely divisible distributions*, vol. 68. Cambridge University Press, New York, 1999.

[75] Schoutens, W. Meixner process in finance. Tech. rep., EURANDOM Report, 2001.

[76] Schoutens, W. *Lévy processes in Finance: Pricing Financial Derivative.* Wiley, 2003.

[77] Schweizer, M. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Anal. Appl. 13*, 5 (1995), 573–599.

[78] Seneta, E. The early years of the variance-gamma process. In *Advances in Mathematical Finance*, M. Fu, R. Jarrow, J.-Y. Yen, and R. Elliott, Eds., Applied and Numerical Harmonic Analysis. Birkh́'auser Boston, 2007, pp. 3–19.

[79] Stace, A. W. *Volume Weighted Average Price Options.* PhD thesis, The University of Queensland, March 2006.

[80] Stace, A. W. A moment matching approach to the valuation of a volume weighted average price option. *The Mathematical Journal 10*, 3 (2007).

[81] Strikerda, J. C. *Finite Difference Schemes and Partial Differential Equations*, 2nd ed. Cambridge University Press, 2005.

[82] Tankov, P. Simulation and option pricing in lévy copula model.

[83] Tankov, P., and Voltchkova, E. Jump-Diffusion models: A practitioner's guide. Technical report, Université Paris VII, September 2006.

[84] Wilmott, P., Howison, S., and Dewynne, J. *Option Pricing: Mathematical Models and Computation.* Oxford Financial Press, 1993.