Honour Thesis Presentation (Mathematics)

The pricing of VWAP Options under Lévy processes framework

Leanne Dong
Supervised by: Professor Alexander Novikov

School of Mathematical Sciences

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Aim

This project aims to

- Describe the dynamics of the stock price using a Lévy process, and the rate of trade using a shifted squared Ornstein-Uhlenbeck process.
- Compute VWAP moments involves computing the moment of a ratio of two random variable.
- Find analytical formulae for moments of VWAP, then approximate a distribution of the VWAP using moment-matching technique.
- Demonstrate the idea of moment-matching technique in pricing option whose payoff depends on both price and trade volume.
- Verify analytical results by Monte Carlo simulation.

Plan of the talk

Existing Literature on VWAP

Literature on VWAP Options: Only 2 paper discuss VWAP options (under the assumption that stock price evolves as a Geometric Brownian Motion)

- Stace (2004) : Moment-matching(ODE/PDE approach)
  - He matched the first two VWAP moments with the Lognormal density by solving a system of 19 ODEs.
  - The solution of the ODEs were verified by MC
  - The solution of the ODEs are then substituted into the expression for the first two Lognormal moments and solved the parameters.
  - He derive and solve the pricing PDE for the VWAP option.
- Novikov-Ling-Kordzakhia (2010) : Moment-matching (Semi-analytical approach)
  - They match the first two VWAP moments with the GIGs(Generalised Inverse Gaussian) density via a semi-analytical approach, which is simpler and fast
  - The analytical VWAP moments were fully verified by MC.
  - The parameters of GIGs are found by matching the analytical VWAP moments to the target distribution moments.
  - VWAP Option price are found using the standard risk neutral valuation technique.

Existing Literature on Lévy processes and Variance Gamma processes

- The Variance Gamma process can be viewed as the difference of two independent Gamma processes.
- The Variance Gamma model (VG) was implemented by Bloomberg to allow analysts to extract a return distribution that takes into account of the implied volatility skew. Deutsche Bank use VG to model credit risk.
- Madan and Seneta (2000) introduced VG model for asset returns, they present a symmetric version of the Variance Gamma process.
- Madan, Carr and Chang (1998) extend the symmetric VG model to an asymmetric form and present a Black-Scholes type formula to price European option under the Variance Gamma process.
- Fu (2000) describes Monte Carlo methods in simulating Variance Gamma process.

Lévy processes : Motivation

Historically

- Continuous-time analogues and limits of random walks, generalisation of Brownian motion
- Prototypes of semimartingales: suitable of stochastic calculus
- Applications to insurance risk theory (ruin problem), storage problems etc.

More recently

- Assets prices do not evolves continuously, they exhibit jumps and spikes.
- Asset returns are not normally distributed, they are fat-tailed and skewed.
- Financial modelling: stochastic volatility, leverage effects
- Risk Management: Credit Risk Modelling

Definition

Definition (Lévy process)

A Lévy process is a continuous-time stochastic process $(X_t, t \geq 0, X_0 = 0)$ *with*

- *Stationary increments*
  $X_{t+s} - X_t \overset{d}{=} X_s$
- *Independent increment* $\forall t_i \in T$ *with* $t_1 < ... < t_n$ *and* $n > 1$
  $X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent random variables*
- *and cádlág paths*

The Structure of Lévy processes

Definition (Infinite divisibility)

A random variable Y (or its distribution) is called infinitely divisible if

$$Y \sim Y_1^{(n)} + ... + Y_n^{(n)}$$

for all $n \geq 2$ and independent, identically distributed $Y_1^{(n)}, ..., Y_n^{(n)}$. The distribution of $Y_j^{(n)}$ will depend on n, but not j

Theorem (Lévy Khintchine: Characteristic function of Lévy process)

A random variable X_t is infinitely divisible iff

$$\psi_X(u) = \exp\left(i\theta u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbb{I}_{|x|<1})\nu(dx)\right)$$

for some $\theta \in \mathbb{R}, \sigma^2 \geq 0$, and ν on $\mathbb{R}/\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$

The Lévy process is usually specified by a Lévy triplet: (θ, σ^2, ν); θ is the deterministic drift, σ^2 is the variance of the diffusion process, ν is the Lévy measure.

Corollary (Lévy decomposition)

The Lévy process can be decomposed into the sum of three parts

$$L_t = \theta t + \sigma W_t + J_t$$

W_t is std BM, J_t is a discontinuous pure jump process Lévy triplet $(\theta, 0, \nu)$. The process W_t and J_t are indepedent.

Time-changed Brownian motion representation

1. Under the VG framework, the log stock price is defined in terms of a Wiener process with drift θ and volatility σ, where t follows a gamma process γ_t^{ν} have mean rate 1 per unit of time and variance ν

$$B_t^{(\theta,\sigma)} \equiv \theta t + \sigma W_t$$

2. This results in the pure jump process that has an infinite number of jumps in any interval of time:

$$X_t^{VG} = B_{\gamma_t^{(\nu)}}^{(\theta,\sigma)} = \theta \gamma_t^{(\nu)} + \sigma W_{\gamma_t}^{(\nu)}$$

3. X_t follows a Variance Gamma distribution.
4. This representation allow one to write down the Characteristic function of VG

$$\psi_{X_t}(u) = \mathbb{E}(e^{iuX_t}) = \left(\frac{1}{1 - iu\theta\nu + (u\sigma)^2 \frac{\nu}{2}} \right)^{\frac{t}{\nu}} \tag{1}$$

Difference-of-gammas representation

- Madan et al. (1998) showed that the VG process may also be expressed as the difference of two independent Gamma processes.
- This characterisation allows the Lévy measure to be determined:

$$\nu_{VG}(dx) = \begin{cases} C\exp(-G|x|)|x|^{-1}dx, & \text{if } x < 0 \\ C\exp(-M|x|)|x|^{-1}dx, & \text{if } x > 0 \end{cases} \quad (2)$$

where

$$\begin{aligned} C &= 1/\nu > 0 \\ G &= 2\mu_+/\sigma_{VG}^2 > 0 \\ M &= 2\mu_-/\sigma_{VG}^2 > 0 \end{aligned}$$

where $\mu_\pm$ are defined in (3)

- With this parametrisation , it is clear that

$$X_t^{VG} = \gamma_t^{(\mu_+,\nu_+)} - \gamma_t^{(\mu_-,\nu_-)}$$

where the two gamma processes are independent (but defined on a common probability space) with parameters

$$\mu_\pm = (\sqrt{\theta^2 + 2\sigma^2/\nu} \pm \theta)/2 \quad (3)$$

$$\nu_\pm = \mu_\pm^2 \nu \quad (4)$$

Adopt the difference-of-gammas representation and define the continuous stock price model as

$$S_t = S_0 e^{L_t}, \quad S_0 > 0 \tag{5}$$

where $L_t = mt + \sigma_{BM} W_t + X_t^{VG}$, assuming 0 interest rate.
In this way, the log-returns of stock prices are no longer Normally distributed.
Note that under the Black-Scholes framework we had

$$\log S_{t+1} - \log S_t \sim Normal\left(\mu_{BM} - \frac{\sigma_{BM}^2}{2}, \sigma_{BM}^2\right)$$

One can move easily from the real world to the risk-neutral one by simply replacing the drift μ with a constant interest rate r (assume no-dividend).

$$S_t = S_0 \exp\left(r - \frac{\sigma^2}{2} t + \sigma W_t\right), \quad t \geq 0$$

In contrast with the Black-Scholes's world; in the case of a Geometric Lévy model, we are working in an incomplete market, meaning that there is no unique transformation. In reality, there are infinite many possible measure change.However, market doesn't has to be complete for the absence of arbitrage to occur, one particular simple transformation is the mean-correcting measure changes, where the Lévy process is shifted in such a way to obtain a martingale.

$$S_t = S_0 e^{L_t} = S_0 e^{mt + \sigma_{BM} W_t + X_t^{VG}}$$

where

$$m = \mu - \frac{1}{2}\sigma_{BM}^2 - \log\left(\frac{1}{1 - \theta v - \frac{1}{2}\sigma_{VG}^2 \nu}\right)/\nu \qquad (6)$$

ensures $e^{-rt}S_t$ is a martingale.

Sketch of the proof

Proposition (Moment generating function of Lévy process)

For any Lévy process $(X_t, t \geq 0)$ *on* $\mathbb{R}$. *There exist a continuous function* $\psi : \mathbb{R} \mapsto \mathbb{R}$ *called the cumulant of* X, *such that*

$$\mathbb{E}(e^{iuX_t}) = \varphi_{X_t}(u) = e^{t\psi(u)} \quad u \in \mathbb{R}$$

$$\Longrightarrow \mathbb{E}S_t = S_0 e^{t\psi(1)} = S_0$$

$$\Longrightarrow \mathbb{E}S_t = S_0 e^{t\psi(1)} = S_0 e^{t(m+\frac{\tilde{\sigma}^2}{2}+\frac{1}{\nu}\log\left(\frac{1}{1-\theta\nu+\frac{(\sigma)^2}{2}\nu}\right))} = S_0$$

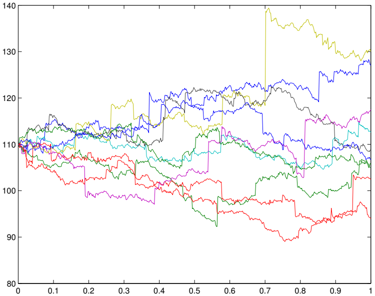
iff

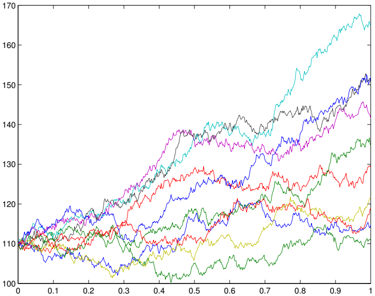
$$m = -\frac{1}{2}\tilde{\sigma}^2 - \frac{1}{\nu}\log\left(\frac{1}{1-\theta\nu+\frac{(\sigma)^2}{2}\nu}\right)$$ assume no growth in stock price

$$m = \mu - \frac{1}{2}\tilde{\sigma}^2 - \frac{1}{\nu}\log\left(\frac{1}{1-\theta\nu+\frac{(\sigma)^2}{2}\nu}\right)$$ assume stock price growths at rate μ

Comparison of Trajectories

The left are the trajectories of a Lévy process with VG jumps, while on the right is a special class of Lévy process, that is the well-known Brownian motion. One the left plot, there are large number of small jumps. The magnitude of the jump progressively concentrate on the origin. In this sense, the VG model respects the intuition underlying the sample path continuity of B.M as a model.





VWAP Options: Definition

VWAP options are options whose payoff depends on a volume weighted average price (VWAP). They were suggested to reduce market manipulation risk. Here we consider VWAP call options.

- Definition (discrete-time)

$$C_T = (A_T - K)^+$$
$$A_T = \sum_{i=1}^{N} w_i S_i$$
$$\sum_{i=1}^{N} w_i = 1, \quad w_i = \frac{U_i}{\sum_{i=1}^{N} U_i}$$

- Definition (continuous-time)

$$C_T = (A_T - K)^+$$
$$A_T = \frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt}$$

- Relation to Asian option: When Volume U_t is constant, The VWAP option is indifferent from an Asian option. $A_T = (\frac{1}{T}\int_0^T S_t dt - K)^+$

To price the option with the payoff function

$$C_T = \left(\frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt} - K \right)^+$$

- $S_t = S_0 \exp\{rt + mt + X_t^{VG} + \sigma W_t\}$ (Stock price evolves as a Geometric Lévy process)
- $U_t = X_t^2 + \delta$ (Trade Volume evolves as a squared Ornstein Uhblenbeck processes)
- $dX_t = \lambda(a - X_t)dt + \nu d\widetilde{W}_t$
- W correlated with $\widetilde{W}$, using the following equations

$$W = \rho \widehat{W} + \sqrt{1-\rho^2}\widetilde{W}$$
$$cov(W, \widetilde{W}) = \sqrt{1-\rho^2}$$

What are the pricing alternatives

- PDE approach: 2 state variables + time variable, leads to intractable solution
  Difficult to formulate boundary condition.
- Probabilistic approach requires a distribution which we do not know.
- Can solve by MC, but slow

Pricing approach: Matching the Lognormal

- Moments of the lognormal process are well known:

$$M_1^{LN} = S_0 e^{\tilde{\mu} t} \qquad M_2^{LN} = S_0^2 e^{2\tilde{\mu} t}(e^{\tilde{\sigma}^2} t - 1)$$

- Calculate the analytical moments M_1 and M_2 for the VWAP, postulate that it represents an effective lognormal process and then match the first two moments:

$$M_1 = M_1^{LN} \qquad M_2 = M_2^{LN}$$

- Express the lognormal moments in function of VWAP moments.

$$\tilde{\mu} = \frac{1}{t} \log\left(\frac{\mathbb{E}(\tilde{S}(t))}{S_0}\right) \tag{7a}$$

$$\tilde{\sigma} = \sqrt{\frac{1}{t} \log\left(\frac{Var(\tilde{S}(t)) + (\mathbb{E}(\tilde{S}(t)))^2}{(\mathbb{E}(\tilde{S}(t)))^2}\right)} \tag{7b}$$

- Moment matching works reasonably well e.g. Asian arithmetic options

Finding the first moment $\mathbb{E}A_T$: $\mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\right)$

The analogous Volume Weighted Average Price (VWAP) is given by

$$A_T = \frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt} \tag{8}$$

Define the Lévy process and the stock price dynamics as

$$S_t = S_0 e^{L_t}$$

$$L_t = mt + \sigma_{BM} W_t + X_t, \quad W_t = \rho\widehat{W} + \sqrt{1-\rho^2}\widetilde{W}, \quad \widetilde{W} \perp \hat{W}$$

The expectation is A_T is computed as the following manner

$$\mathbb{E}(A_T) = \mathbb{E}\left(\frac{\int_0^T S_t U_t dt}{\int_0^T U_t dt}\right)$$

$$= \int_0^T \mathbb{E}\left(\frac{S_t U_t}{V_T}\right) dt = \int_0^T \mathbb{E}\left(\frac{S_0 e^{mt+\sigma_{BM}(\rho\widehat{W}_t+\sqrt{1-\rho^2}\widetilde{W}_t)+X_t} U_t}{\int_0^T U_t dt}\right) dt$$

$$= S_0 \int_0^T \mathbb{E}e^{X_t+mt+\sigma_{BM}\sqrt{1-\rho^2}\widetilde{W}_t} \mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\right) dt$$

$$\mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\right) = e^{\frac{1}{2}\rho^2\tilde{\sigma}^2 t}\widetilde{\mathbb{E}}\left(\frac{U_t}{V_T}\right) \quad \text{is wanted}$$

This means that we need a Radon-Nikodym derivative along the lines.

Finding the first moment $\mathbb{E}A_T$: $\mathbb{E}\left(\frac{e^{\sigma BM \rho W_t} U_t}{V_T}\right)$

Theorem (Girsanov)

Let W_t be a $\mathbb{P}$ Brownian Motion, and let λ_t be an adapted process satisfying $\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T |\lambda_t|^2 dt\right)\right) < \infty$, moreover let $\mathcal{E}_T(\cdot)$ be the Doléan exponential. Define an equivalent measure $\widetilde{\mathbb{P}}$ by

$$\begin{aligned}\eta_t = \left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} &= \mathcal{E}_t\left(\int_0^\cdot \lambda_s dW_s\right) \quad \mathbb{P} \quad a.s. \\ &= \exp\left(\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t |\lambda_s|^2 ds\right) \quad \forall t \in [0,T]\end{aligned}$$

Then the process

$$\widetilde{W}_t = W_t - \int_0^t \lambda(s) ds, \quad t \in [0,T] \quad \text{is a standard B.M. under } \widetilde{\mathbb{P}}$$

Finding the first moment $\mathbb{E}A_T$: $\mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widetilde{W}_t}U_t}{V_T}\right)$

Define the Radon Nikodym derivative as

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = e^{\rho\sigma_{BM}\widehat{W}_t - \frac{\rho^2}{2}\sigma_{BM}^2 t}$$

is an exponential martingale and therefore by Girsanov theorem,

$$d\widetilde{W}_t = dW_t + \rho\sigma_{BM}dt$$

$$\begin{aligned}\mathbb{E}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\right) &= \widetilde{\mathbb{E}}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}\left.\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right|_{\mathcal{F}_T}\right)\\ &= \widetilde{\mathbb{E}}\left(\frac{e^{\sigma_{BM}\rho\widehat{W}_t}U_t}{V_T}e^{-\rho\sigma_{BM}\widehat{W}_t + \frac{\rho^2}{2}\sigma_{BM}^2 t}\right)\\ &= e^{\frac{1}{2}\rho^2\tilde{\sigma}^2 t}\widetilde{\mathbb{E}}\left(\frac{U_t}{V_T}\right)\end{aligned}$$

Proposition

$$\mathbb{E}A_T = \int_0^T e^{(\mu+\frac{1}{2}\rho^2\sigma_{BM}^2)t}\widetilde{\mathbb{E}}\left(\frac{U_t}{V_T}\right)dt$$

where $\widetilde{\mathbb{E}}(\cdot)$ is the expectation with respect to measure $\widetilde{\mathbb{P}}$.

$$V_T = \int_0^T U_t dt$$

Finding the first moment $\mathbb{E}A_T$: $\mathbb{E}\left(\frac{U_t}{\int_0^T U_t dt}\right)$

The problem now reduced to compute $\mathbb{E}\left(\frac{U_t}{\int_0^T U_t dt}\right)$, having granted the idea from Novikov-Ling-Kordzakhia(2010), This expectation can be found by computing the Laplace transform of the integral of the squared Ornstein-Uhblenbeck process. Calculations of this kind involves measure changes again.

Defined the joint Laplace transform as

$$\phi(z,r,q) = \mathbb{E}(\exp\{-zU_t - rU_s - qV_T\}) \tag{9}$$

Assume

$$\mathbb{E}(U_t/V_T) < \infty$$

$$\begin{aligned}
\mathbb{E}\left(\frac{U_t}{\int_0^T U_t dt}\right) &= -\frac{\partial}{\partial z}\bigg|_{z=0} \int_0^\infty \mathbb{E}(e^{-zU_t - q\int_0^T U_t dt})dq \\
&= -\frac{\partial}{\partial z}\bigg|_{z=0} \int_0^\infty \phi(z,0,q)dq \\
&= -\int_0^\infty \frac{\partial}{\partial z}\bigg|_{z=0} \phi(z,0,q)dq
\end{aligned} \tag{10}$$

Finding the first moment $\mathbb{E}A_T$: $\mathbb{E}\left(\frac{U_t}{\int_0^T U_t dt}\right)$

Now find $\phi(z,0,q)$

$$\begin{aligned}
\phi(z,0,q) &= \mathbb{E}(\exp(-zU_t - q\int_0^T U_t dt)) \\
&= \mathbb{E}(\exp(-z(X_t^2+\delta) - q\int_0^T (X_t^2+\delta)dt)) \\
&= \exp\{-z\delta - q\delta T\}\,\psi(z,q) \\
&= \exp\{-z\delta - q\delta T\}\exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\gamma(z,q) \\
&= \exp\{-z\delta - q\delta T\}\exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\mathbb{E}\left(\mathbb{E}\left(\exp\left\{-2zvaY_t - 2qva\int_0^T Y_s ds - zv^2Y_t^2 + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\Big|\,\mathcal{F}_t\right)\right) \\
&= \exp\{-z\delta - q\delta T\}\exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\mathbb{E}\left(e^{\xi}\mathbb{E}\left(\exp\left\{-2qva\int_t^T Y_s ds + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\Big|\,\mathcal{F}_t\right)\right) \\
&= \exp\{-z\delta - q\delta T\}\exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\mathbb{E}\left(e^{\xi}\mathbb{E}\left(\exp\left\{-2qva\int_t^T Y_s ds + \frac{(\kappa-\lambda)Y_T^2}{2}\right\}\,\Big|\,\mathcal{Y}_t\right)\right) \\
&= \exp\{-z\delta - q\delta T\}\exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\mathbb{E}\left(\exp\left\{-2qvX_1 + \frac{(\kappa-\lambda)X_2^2}{2}\right\}\,\Big|\,X_3 = z\right) \\
&= \exp\{-z\delta - q\delta T\}\exp\left\{-za^2 - qa^2T + \frac{(\lambda-\kappa)T}{2}\right\}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-2qvx + \frac{(\kappa-\lambda)y^2}{2}\right\} f_{X_1,X_2|X_3}(x,y\mid z)\,dxdy \\
&= \exp\{-z\delta - q\delta T\}\exp\{\underbrace{HY_t^2 + JY_t + L}_{\text{some constant}}\}
\end{aligned}$$

Note

- $\psi(z,q) = \mathbb{E}(\exp\{-zX_t^2 - q\int_0^T X_t^2 dt\})$,the elimination of $\int_0^T X_t^2 dt$ is done via measure(see [1])
- $\gamma(z,q) = \mathbb{E}(\exp\left\{-2zvaY_t - 2qva\int_0^T Y_s ds - zv^2Y_t^2 + \frac{(\kappa-\lambda)Y_T^2}{2}\right\})$ can be computed using the tower law of conditional expectation and the markov property of standard OU process Y_t

The Second moment $\mathbb{E}(A_T^2)$

Using the same measure change technique, the VWAP second moment is not hard to find. So we have the following proposition

$$\mathbb{E}A_T^2 = \int_0^T \int_0^T e^{(\mu+\frac{1}{2}\rho\sigma_{BM}^2)(t+s)+(s\wedge t)(\sigma_{BM}(1-\rho^2)+2\log(1-\theta\nu+\frac{\sigma_{VG}^2}{2}\nu)/\nu)-\log(1-2\theta\nu+2\sigma_{VG}^2\nu)/\nu)} \mathbb{E}\left(\frac{U_t}{V_T^2}\right)$$

Proposition

$$\mathbb{E}A_T^2 = \int_0^T \int_0^T e^{(\mu+\frac{1}{2}\rho\sigma_{BM}^2)(t+s)+(s\wedge t)(\sigma_{BM}(1-\rho^2)+2\log(1-\theta\nu+\frac{\sigma_{VG}^2}{2}\nu)/\nu)-\log(1-2\theta\nu+2\sigma_{VG}^2\nu)/\nu)} \mathbb{E}\left(\frac{U_tU_s}{V_T^2}\right)$$

Given the Laplace Transform $\phi(z,r,q)$ in (9) where

$$\mathbb{E}\left(\frac{U_tU_s}{V_T^2}\right) = \int_0^\infty q\frac{\partial}{\partial z}\frac{\partial}{\partial r}\Phi(z,r,q)\Big|_{z=r=0} dq \tag{11}$$

Note that the correlation make no impact on the computation of the first moment under both models.

Remark

$$\mathbb{E}A_T^{VG} = \mathbb{E}A_T^{GBM} = S_0e^{\mu t}$$

Table: Numerical values of $\mathbb{E}(A_T)$ and Monte Carlo simulation of $\mathbb{E}(A_T)$ for varying stock price volatility value σ_{BM}.

σ_{BM}	$\mathbb{E}(A_T)$ [a]	$\hat{\mathbb{E}}(A_T)$	MC std. error	Rel.error (%)
0.1	115.68	115.67	0.007	0.0095
0.2	115.68	115.69	0.0138	0.0078
0.3	115.68	115.66	0.0203	0.018
0.4	115.68	115.65	0.0273	0.027
0.5	115.68	115.67	0.0078	0.053

Note: take $\rho = 0$, drift, symmetry,kurtosis,skewness parameters are held constant as $\mu = 0.1, \sigma = 0.1, \theta = 0.14, v = 0.1$ and $\sigma_{VG} = 0.1$

[a] Note that σ_{BM} does not enter into the computation of $\mathbb{E}(A_T)$, which leads to unchanging values for this column

Table: Numerical values of $\mathbb{E}(A_T^2)$ and Monte Carlo simulation of $\mathbb{E}(A_T^2)$ for varying stock price volatility value σ_{BM}.

σ_{BM}	$\mathbb{E}(A_T^2)$	$\hat{\mathbb{E}}(A_T^2)$	MC std. error	Rel.error (%)
0.1	13402.8	13403.7	0.007	0.0067
0.2	13531.2	13548.3	0.0138	0.1300
0.3	13766.7	13760.3	0.0203	0.04649
0.4	14106.7	14102.2	0.0273	0.0319
0.5	14562.3	14564.4	0.0078	0.0144

Valuation of VWAP Options

Consider a VWAP with expiry date T, struck at K, consider as terminal payoff function

$$C_T = (A_T - K)^+$$

By the previous postulation that the process of

$$\widetilde{A}_T \sim lognormal(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2, \tilde{\sigma}^2) \Longleftrightarrow \widetilde{A}_T = e^{\tilde{\sigma}W_1 + \tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2}, \quad W_1 \sim N(0,1).$$

Via the moment-matching technique, we match the first two VWAP moments to the well-known Log normal distribution

$$\mathbb{E}A_T = \mathbb{E}(\widetilde{A}_T) = e^{\frac{1}{2}\tilde{\sigma}^2 + \tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2} = e^{\tilde{\mu}}$$

$$\mathbb{E}A_T^2 = \mathbb{E}(\widetilde{A}_T^2) = e^{2\tilde{\sigma}^2 + 2(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)} = e^{2\tilde{\mu} + \tilde{\sigma}^2}$$

Assuming the terminal VWAP price is lognormal, the price of the VWAP call struck at K, expiring at T can be calculated as the expected discounted payoff

$$\ln \widetilde{A}(T) \sim N(\ln \widetilde{A}(0) + (\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)T, \tilde{\sigma}^2 T)$$

$$C_0 = \mathbb{E}(e^{-rT}(\widetilde{A}_T - K)^+) = \int_{\ln K}^{\infty} e^{-rT}(e^z - K)\mathbb{I}_{\{e^z > K\}} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(z - \ln S(0) - (r - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}} dz$$

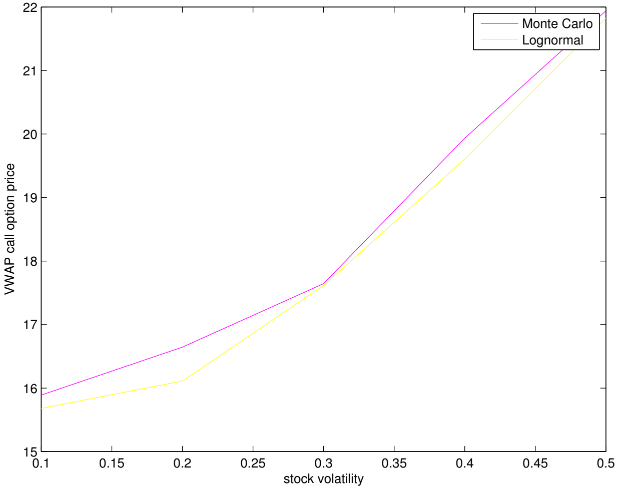
Comparison of call price

Input parameters: $\tilde{S}(0) = 110, K = 100, r = 0, T = 1$

Table: Numerical values of call price and Monte Carlo simulation of call price for varying stock price volatility value σ_{BM}.

σ_{BM}	MC	Lognormal
0.1	15.89	15.68
0.2	16.64	16.11
0.3	17.64	17.61
0.4	19.94	19.61
0.5	21.94	21.82

- Given stock price volatility as a input, lognormal parameters are solved and options prices are computed by integrating the terminal payoff against the Gaussian density, using *NIntegrate* function in *Mathematica*.
- As the second moments are different for different stock price volatility, the lognormal parameter $\tilde{\sigma}$ differs when varying stock price volatility, as a consequence, the price of the VWAP option is different for different stock price volatility.


Monte Carlo
Lognormal
VWAP call option price
stock volatility
22
21
20
19
18
17
16
15
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Summary

- Have an explicit formula of analytical moments of the VWAP under the assumption that volume process is driven by a shifted squared O-U and asset price is driven by a Geometric Lévy process.
- Have a way to address the dependence between the stock price and trade volume.

Future Work

- Find more moments, use other expansions such as Gram Charlier Expansion
- Identify better distribution to match
- Better(Faster) Monte Carlo Algorithm
- Hedging issues

Bibliography

A.A. Novikov, T.G. Ling, and N. Kordzakhia.
Pricing of volume-weighted average options: analytical approximation and numerical results.
unspecified, 2010.