# Stochastic Navier-Stokes equations on 2D rotating spheres with stable Lévy noise

Leanne Dong

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# Abstract

The aim of this dissertation is to study stochastic Navier-Stokes equations (SNSE) on 2D rotating spheres in Hilbert space perturbed by pure jump Lévy noise of $\beta$-stable type. The first goal is to establish the well-posedness of solutions to this class of equations. The second goal is to investigate qualitative questions on ergodicity, asymptotic behaviour and random dynamics. In Chapter 2, we review the analytic and probabilistic preliminary required to present the main results of the thesis. Then we introduce the background material on Hilbert space valued cylindrical Lévy noise via subordination of $\beta$-stable type. In Chapter 3, we prove the existence and uniqueness of solutions to the SNSE under suitable assumptions of noise and forcing and, in the second part, we deduce the existence of an invariant measure with measure support. Chapter 4 is devoted to the study of random dynamical systems generated by our SNSE. In particular, we prove that, with sufficient regularity, there exists a finite-dimensional random attractor for our SNSE. Moreover, such a random attractor supports a Feller Markov Invariant measure.

# Statement of Authorship

I certify that to the best of my knowledge, the content of this thesis is my own work and that all the assistance received in preparing this thesis and sources have been acknowledged. I also certify that this thesis has not been submitted for the award of any other degree or diploma.

Leanne Dong, April 19, 2018

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# Preface

This thesis is concerned with stochastic Navier-Stokes equations on the 2D rotating unit sphere, with external force perturbed by an additive stable Lévy noise. (It is called SNSE in short.)
The goal is to perform a systematic analysis of rotating stochastic fluid with Lévy noise on spheres, in attempting to extend the work initiated by Goldys et al. [14, 20] under Wiener noise.
The approach taken here is functional analytic and measure-theoretic. This means that the SNSE is interpreted as a random nonlinear PDE in a Hilbert space $(H, |\cdot|)$. We do not address in this thesis any computational aspects, despite the presence of this problem in our thoughts; neither do we address control-theoretic questions or fluid mechanical problems.
In Chapter 1, we provide a comprehensive survey of Stochastic Navier-Stokes equations with Lévy noise. We aim to summarise, using a homogeneous notation, the main ideas, and approaches in the literature of SNSE which relates to our studies.
In Chapter 2, we start with a collection of some known definitions and theorems that we will make use of in the rest of the thesis. This material is necessary for the development in later chapters. Then we give an overview of the theory of Cylindrical Lévy processes, then we investigate their stochastic Integrals and present a new integration (Fubini) theorem.
Chapter 3 was devoted to the issue of well-posedness and invariant measure. Namely, the existence and uniqueness of the solution and its continuous dependence on the initial data. In particular, we prove the existence of a weak solution via Galerkin approximation, then following standard arguments in the classical theory of NSE in the spirit of Lion and Prodi [74], Temam [10], the weak solution is proved to be unique. Moreover, the solution is shown to depend continuously on the initial datum. Furthermore, following a semigroup method in Brzezniak and Capinski [19], the solution is shown to be strong indeed. Finally, we deduce the existence of invariant measures and establish that an invariant measure is supported in $V$.
The results deduced in Chapter 3 allow us to study the random dynamics of our SNSE in Chapter 4. Under suitable conditions, we prove that our stochastic Navier-Stokes system generates a random dynamical system. Moreover, we prove that the generated random dynamical system has a finite-dimensional compact random attractor. Finally, we show that the attractor carries an invariant Markov measure.
In Chapter 5 we provide a conclusion and discuss a summary of our contribution and limitation of our work. We also discuss possible future extensions of the line of research in this thesis.

# CHAPTER 1

## Introduction

### 1.1 Motivation

In this thesis, we initiate a systematic study of well-posedness, invariant measures and asymptotic behaviour for stochastic Navier-Stokes with Lévy Noise of stable-type on a 2D rotating sphere. This thesis is motivated by the existing studies of the following five areas:

- Navier-Stokes Equations on spheres
- Stochastic Navier-Stokes Equations (SNSE) in 2D bounded smooth domain
- Stochastic PDE perturbed by Lévy Noise
- SNSE perturbed by Lévy Noise in 2D bounded smooth domain
- Asymptotic behaviour of SNSE and the notion of random attractor

*Why would one study Navier-Stokes equations on spheres?*
Problems arising in meteorology naturally reduce to fluid equations on a 2D manifold. The deterministic Navier-stokes equations (NSE) on the sphere serves as a powerful tool in modeling geophysical flow ([82, 83]) and has been an object of intensive study since 1990. Many authors have studied the NSE on the unit sphere. Notably, Il'in and Filatov [64, 66] considered the existence and uniqueness of solutions to these equations and the estimation of the Hausdorff dimension of their global attractors [65]. Teman and Wang studied the inertial forms of NSEs on the sphere while Teman and Ziane proved that the NSE on a 2D sphere is a limit of NSE defined a spherical cell [104]. In another direction, Cao, Rammaha and Titi proved the Gevrey regularity of the solution and found an upper bound on the asymptotic degrees of freedom for the long-time dynamics [24].
*Why Stochastic Navier-Stokes Equations?*
The Navier-Stokes equations subject to random perturbation, such as white noise, can be used as a model to explain the random fluctuations observed in the velocity profile of viscous incompressible fluid flow. Such a perturbed system is described as a nonlinear stochastic system known as stochastic Navier-Stokes equations (SNSE). Stochastic analysis of such equations allows one to answer certain difficult problems in hydrodynamics and provides insights into the

modeling of turbulence. Stochastic Navier-Stokes equations, as statistical models for turbulent fluid motion, have been intensively studied in the last 20 years. For instance, existence, uniqueness and ergodicity have been studied by many authors under perturbation with Wiener noise. It is well known in deterministic theory that there exists a unique global (in time) strong solution in dimension two. (See Kiselev and Ladyzhenskaya [67] for bounded domain, [71]). Much of the results in deterministic theory have been generalized to the stochastic case with Wiener perturbation and now it is well known that SNSE has a unique global strong solution in dimension two, comparable to the deterministic case.

*Why Lévy Processes?*

Lévy motions, particularly non-Gaussian process, have been widely applied to Biology, Image processing, Climate forecast and certainly in Finance and Physics [56, 59, 87, 98, 117]. From a fluid modeling point of view, although continuous models are good enough in a macroscopic scale, at an atomic scale, the model breaks down, and the use of a Lévy process is compelling as fluid is not continuous at a microscopic scale [76]. As a special non-Gaussian stochastic process, the stable-type process attracts more and more mathematical interests due to the properties which the Gaussian process does not possess. The tail of Gaussian random variable decays exponentially which does not fit well for modeling processes with high variability or some extreme events, such as earthquakes or stock market crashes. In contrast, the stable Lévy motion has a 'heavy tail' that decays polynomially and can be useful for these applications. For instance, when heavier tails (relative to a Gaussian distribution) of asset returns are more pronounced, the asymmetric $\alpha$-stable distribution becomes an appropriate alternative in modeling [88].

In recent years, stochastic equations driven by Lévy type noise have attracted much attention. The study of SNSE is now trending toward perturbation with Lévy noise. The study is motivated greatly by engineering applications. For instance, modeling aerodynamical flow subject to abrupt disturbance due to climate change [93]. It was proved in [41] that there exists an invariant measure for the 2D SNSE with Lévy noises of square integrable type. Most publications up to date rule out the interesting $\alpha$-stable case. To our knowledge there is only one publication that discusses SNSE with Lévy noise of stable type (see [41]).

The study of the asymptotic behaviour of dynamical system is one of the central problems in Mathematical Physics. One way to attack the problem for deterministic dynamical systems is to find conditions for the existence of an attractor, which is a compact set in the phase space which determines the long-time behavior of the dynamical system. The theory of attractors for the deterministic infinite dynamical systems is well established. This line of research was first introduced by [68] and the theory has now been generalised to the stochastic case with Gaussian noise. In this direction, the notion of random attractors was first introduced for random PDE by Brzeźniak, Capiński and Flandoli [19]. Later, it was generalised to the case of Itô equations (stochastic PDE) by Crauel and Flandoli [34]. In their work, the authors proved the existence of a global attractor for the SNSE in a 2D bounded domain with sufficient regular noise. Following this major breakthrough, the study of random attractors has gained considerable attention during the past decade. A comprehensive survey is presented in [7]. In this thesis, we generalise the line of research of random attractors in the spirit of Crauel and Flandoli to the case of discontinuous noise.

## 1.2 A short introduction to stochastic Navier-Stokes equations with Lévy noise

In this thesis, we consider the following stochastic Navier-Stokes equations (SNSE) describe the motion of an incompressible rotating fluid on a 2D unit sphere subject to random external force:

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \nabla p = f + \eta(x,t), \quad \operatorname{div} u = 0, \quad u(x,0) = u_0 \tag{1.1}$$

where $\mathbf{L}$ is the stress tensor, $\omega$ is the Coriolis acceleration, $p(t,x)$ is the pressure of the fluid, $f$ stands for the deterministic external force and $\eta$ is the Lévy white noise which can be informally viewed as the derivative of an $H$-valued Lévy process. The vector field $u(t,x) = (u_\theta(t,x), u_\phi(t,x))$ is the velocity, $\nu$ is the viscosity. The differential operators $\nabla$ and div are the surface gradient and surface divergence. Rigorous definitions of all the quantities in this equation are given in in Chapter 3.
The nonlinear term $\nabla_u u(x)$ is defined as

$$\pi_x \left( \sum_{i=1}^{3} \tilde{u}_i(x) \partial_x \tilde{u}(x) \right) = \pi_x \left( (\tilde{u}(x) \cdot \tilde{\nabla}) \tilde{u}(x) \right) \quad x \in \mathbb{S}^2,$$

where $\pi_x : \mathbb{R}^3 \to T_x\mathbb{S}^2$ of $x$ onto $T_x\mathbb{S}^2$ is the orthogonal projection. The vector field $\tilde{u}$ relates to $u$ as

$$\tilde{u} = u + u^\perp, \quad u \in T_x\mathbb{S}^2, \quad u^\perp = (u \cdot x)x.$$

By adding a Lévy white noise term, we obtain the SNSE on the sphere:

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \nabla p = f + \eta(x,t),$$

where $\eta = G\frac{d\tilde{L}}{dt}$, $G : H \to H$ is a bounded linear operator.

We introduce the spaces

$$H := \{u \in \mathbb{L}^2(\mathbb{S}^2) : \nabla \cdot u = 0\}$$
$$V := H \cap \mathbb{H}^1(\mathbb{S}^2)$$

where $\mathbb{H}^1(\mathbb{S}^2)$ is the Sobolev space of vector fields on $\mathbb{S}^2$. The inner product of $H$ is the same as $\mathbb{L}^2(\mathbb{S}^2)$ and is denoted as $(\cdot,\cdot)$ and the corresponding norm is $|\cdot|$.
Applying the Leray-Helmholz projection, (1.1) can be recasted into an abstract evolution equation in $H = L^2(\mathbb{S}^2)$ without the pressure term:

$$du(t) + Au(t)dt + B(u(t), u(t))dt + Cu = fdt + GdL(t), \quad u(0) = u_0, \tag{1.2}$$

where $f$ is the deterministic forcing and $u_0$ is the initial velocity. The operator $A$ is defined as

$$A : D(A) \subset H \to H, \quad A = -P(\mathbf{\Delta} + 2\mathrm{Ric}), \quad D(A) = \mathbb{H}^2(\mathbb{S}^2) \cap V. \tag{1.3}$$

We set

$$B(u, v) = \pi(u, \nabla v), \quad B(u) = B(u, u).$$

The operator $\mathbf{C} = P\mathbf{C}_1$ is well defined and bounded in $H$ and $\mathbf{C}_1$ is the Coriolis operator $\mathbf{C}_1 : \mathbb{L}^2(\mathbb{S}^2) \to \mathbb{L}^2(\mathbb{S}^2)$ is defined by the formula

$$(\mathbf{C}_1 v)(x) = 2\Omega \cos\theta(x \times v(x)), \quad x \in \mathbb{S}^2.$$

### 1.2.1 The linear equation

In the linear case and in the absence of deterministic external force, the equation has the form

$$dz + Az + Cz = GdL(t), \quad z(0) = 0. \tag{1.4}$$

Using classical argument (See [29]), one can show there exists unique solution to (1.4), given by

$$z(t) = \int_0^t e^{-(t-s)(A+\mathbf{C})} GdL(t).$$

Using Lemma 3.2.7 in Chapter 4, one can show

$$z \in L^4(0, T; \mathbb{L}^4(\mathbb{S}^2)) \quad \mathbb{P} \quad \text{a.s.}$$

## 1.3 Literature, outline of Chapters and Contributions

In this thesis, we answer the following questions:

- Existence and uniqueness of (weak and strong) solutions of (1.1) and continuous dependence on initial data;

- Existence of invariant measures of (1.1);
- Existence of a finite dimensional random attractor of (1.1).

By removing the noise term $\eta(x,t)$ in (1.1), one obtains the deterministic Navier-Stokes equations describing the incompressible flow past a 2D sphere. These equations are extensively studied dates back to the 90's. Notably, Il'in and Filatove[64] proved there exists unique generalised solutions using the Galerkin method. By letting viscosity $\nu$ tends to 0, they obtain the solution of the Euler equation in the limit. Moreover, the Hausdorff dimension of the global attractor is estimated. Temam and Wang [106] established the existence of an inertial form of the equations. They also studied the associated reaction-diffusion system and found conditions for the spectral gap. For the stochastic counterpart, the questions of existence and uniqueness, and continuous dependence on initial data have been thoroughly investigated in Goldys et al. [14] where $\eta(x,t)$ is a Gaussian white noise. In the same publication, the existence of invariant measures is also deduced from the well-posedness result. The existence of random attractor for (1.1) is proved in a recent publication [15]. In particular, they assume the driving noise is given as an infinite dimensional Wiener process. In contrast, most existing works on random attractors assume finite dimensional noise.

SNSE with Wiener noise has been intensively studied. See for instance the classical publications, Bensoussan and Temam [10], Bensoussan [9], Flandoli and Gatarek [53] , Sritharan and Menaldi [78]. For books, readers are referred to Vishik and Fursikovn [112], Capinski and Cutland [25].

SNSE driven by Lévy noise attracts much attention in recent years. The question of existence and uniqueness of solutions to the SNSE has been thoroughly investigated in 2D domain (bounded and unbounded). For strong solution, see [39, 50]. For Mild solutions, see [49]. For Martingale solutions, see [43, 93].

Dong and Xie [39] study the following two forms of SNSE driven by additive Poisson noise:

$$\begin{cases} \frac{d}{dt}X(t) + \nu AX(t) + BX(t) = \int_U f(X(t-),u)\tilde{N}(dt,du), \\ X(0) = x, \end{cases} \tag{1.5}$$

and additive Poisson and nondegenerate Wiener noise on 2D Torus:

$$\begin{cases} \frac{d}{dt}X(t) + \nu AX(t) + BX(t) = \int_U f(X(t-),u)\tilde{N}(dt,du) + \sqrt{Q}dW(t), \\ X(0) = x, \end{cases} \tag{1.6}$$

where $U$ is a cylindrical Wiener process with covariance operator $I$, $W_t$ is a cylindrical Wiener process with covariance operator $I$, $Q$ is a nuclear operator on the space $H$. The term $\tilde{N} := N(ds,du) - \lambda(du)ds$ is a compensated Poisson random measure, where $\lambda(du)$ is a $\sigma$-finite Lévy measure with an associated Poisson random measure $N(ds, du)$ such that

$$\mathbb{E}N(ds, du) = \lambda(du)ds.$$

Moreover, the process $W(t)$ and $\tilde{N}(ds, du)$ are mutually independent.
By fixing some measurable subset $U_m$ of $U$ with $U_m \uparrow U$ and $\lambda(U_m) < \infty$. Under the hypothesis
**Hypothesis 1** There exists positive constants $C$, $K$ such that

- $\int_U |f(0, u)|^2 \lambda(du) = C < \infty$;
- $\int_U |f(x, u) - f(y, u)|^2 \lambda(du) \leq K|x - y|^2$;
- $\sup_{|x| \leq M} \int_{U_k^c} |f(x, u)|^2 \lambda(du) \downarrow 0$ as $k \uparrow \infty$

Dong and Xie proved that there exists a unique global strong (pathwise) solution for the equation (1.5) and a unique weak solution to (1.6), using the Galerkin approximation argument. Existence of invariant measures is also established for both equations. More precisely, this existence theorem states that,
**Theorem 3.3** [39] Suppose that Hypothesis 1 holds.

(i) For the initial value $x \in H$, both (1.5) and (1.6) have unique weak solutions $X$ in $H$.
(ii) Moreover, if $f \in V \times U \to V$ is measurable, then, if the initial value $x \in V$, (1.5) has a unique strong (pathwise) solution in $D(A)$ after the first jump time of $N(ds, du)$.
(iii) There exists an invariant probability measure for $X(t)$ which is the solution of (1.5) and (1.6) which is loaded on $V$.

In comparison to our studies, we use cylindrical stable noise as the noise term, so all three assumptions in Hypothesis 1 cannot be satisfied as $\int_U u^2 \lambda(du)$ could be infinite. Hence we have to use a different argument to tackle well-posedness, which will be detailed in Chapter 3.
The existence and uniqueness result in [39] is further generalised to 2D unbounded domain case. In particular, Fernando and Sritharan [50] consider the SNSE with jump noise and multiplicative white noise. They proved existence and uniqueness of a strong (pathwise) solution. The difficulty in unbounded domain is the lack of compactness. To overcome this difficulty, they use a method developed in Sritharan and Menaldi [78] for unbounded domain in Gaussian case[78], which exploits the local monotonicity of stokes operator and nonlinear term, and used a modified Minty-Browder technique.
Sritharan et al.[49] study mild solutions of the SNSE with additive noise given as a compensated Poisson random measure of the following form:

$$\begin{aligned} du(t) &= -[Au(t) + B(u(t))] + \int_Z \phi(x, s)\tilde{N}(dt, dx) \\ u(0) &= u_0. \end{aligned}$$

in $L^p$ space, for $p > m$, $m \geq 2$. Where $Z$ is a separable Banach space and $\phi$ belongs to the set of progressively measurable functions, which are square integrable w.r.t. the Lévy measure

$\nu$. The main result in [49] is Theorem 3.2, which proves the existence and uniqueness of a local mild solution to a stochastic Navier-Stokes equation with jump noise in the function space $L^\infty([0,T],J_p)$. Here $J_p$ is a separable Banach space with $L^p(\mathbb{R}^n)$ norm. The crux of the proof is an estimate of the nonlinear term $(u\cdot\nabla)u$ which derived using the properties of the fractional power of the Stoke operator (See Giga and Miyakawa [58]) and the rich mathematical properties of the compensated Poisson random measure. Theorem 3.2 generalises the earlier results of $L^p$ theory of local solutions for deterministic NSE. This work is subsequently generalised to the case $p \geq m$, $p \in [m,\infty)$. See the recent publication [79].
Under the same framework in [39], Dong and Xie [40] proved the ergodicity of the two-dimensional Navier-Stokes equation perturbed by Lévy noise with Wiener term: (1.6), under the nondegeneracy assumption of noise covariance operator $Q$:

- $Q : H \to H$ is a bounded linear operator, injective, with range $R(Q)$ dense in $D(A^{\frac{1}{4}+\frac{\alpha}{2}})$ and $D(A^{2\alpha}) \subset R(Q) \subset D(A^{\frac{1}{4}+\frac{\alpha}{2}+\varepsilon})$.

The main contribution of the publication [40] is the proof of strong Feller property, which relies on some a priori estimates on $D(A^\alpha)$, $\alpha \in [1/4, 1/2)$, a stopping time technique (These were used in [51]), and the Bismut-Elworthy-Li formula.
Sritharan and Mohan [79] consider the controlled stochastic Navier-Stokes equation with jump noise and multiplicative white noise in a two-dimensional bounded domain. Using a semi-martingale formulation, the authors seek to find a solution to the martingale problem as well as the associated probability law. The existence and uniqueness of invariant measures follow closely to the ergodic results of the uncontrolled counterpart, which already been studied in [40]. Then it is established that, for the controlled SNSE driven by Lévy noise, it is possible to choose a stationary control corresponds to a statistically stationary turbulent state at the minimum cost functionals.
One must point out that, the method to prove ergodicity in [40] is not applicable in our case in this thesis in obtaining strong Feller and irreducibility, as the three conditions outlined in Hypthesis 1 are not satisfied in the case of stable noise.
Regarding the study of SNSE with stable type Lévy noise, the only result to our knowledge is the work [41] due to Dong, Xu and Zhang. The authors consider the stochastic 2D Navier-Stokes equations on the torus driven by an infinite-dimensional cylindrical Lévy process (in particular the stable process). Under some assumptions on the Lévy measure of the noise, the well-posedness of the problem is established. In more detail, they constructed weak solution using Galerkin approximation, and deduced moment a priori estimate and the existence of probability law using a version of Itô foruma for Lévy process (see p.226 in [5]). Pathwise uniqueness was proved using classical method of the 2D NSE. Then, invoking the famous Yamada-Watanabe theorem, the uniqueness of weak solution (in probabilistic sense) is established. Morever, the

authors proved the existence of invariant measures by using the classical Krylov-Bogolyubov argument. But the uniqueness of the invariant measure has remained open.
In comparison to our result of weak solutions in Chapter 3, we also use Galerkin method to construct a weak (pathwise) solution. However our approach is functional analytic and the existence result is global indeed. Based on our specific assumption on noise $z \in L^4([0,T]; L^4(\mathbb{S}^2) \cap H)$, we deduce a priori estimates (using standard PDE argument) which yields global existence.
The Navier-Stokes equations driven by the compensated Poisson random measure in 3D bounded domains were also studied. Dong and Zhai [43] consider the martingale problem associated to the SNSE, i.e., a solution defined as a probability measure satisfying suitable conditions. In 2013, the existence results on Lévy noise were generalised to the critical case of 2D and 3D unbounded domains by Motyl [80].
Finally, it worth mentioning that there is one publication (due to Varner [109], see also his PhD thesis [110]) discuss the ergodicity of SNSE on the sphere under Gaussian kick-force. In particular, the author proved the existence and uniqueness of an invariant measure for the kick-forced Navier-Stokes system on the 2D sphere, first without deterministic force and then with a time-independent deterministic force. The analogous result for the white noise forced Navier-Stokes system on the 2D sphere without a deterministic forcing is also shown. Furthermore, it was shown the measure is supported in the full space of admissible vector fields of the sphere. One must point out that this approach is quite different from the approach we have taken in this thesis, as we will see in subsubsection 1.3.3.

In the case where noise is given by stable Lévy noise on the sphere, all three questions have remained open. These questions will be addressed in this thesis. In other words, this is the first thesis on stochastic Navier-Stokes equations with Lévy noise on the rotating sphere. There has been a few recent contributions (see for instance [14, 15, 109]) concerned with the similar equations on the 2D rotating sphere but perturbed by Gaussian noise. We review some of these which inspire our study of weak solutions, invariant measures and the Random dynamical system generated by the stochastic Navier-Stokes equations with Lévy noise in Chapter 3 and 4.
Our plan in the remainder of this section is to sketch the main ideas and approaches in some related studies relate to this thesis. We do not enter too much detail and refer readers to the original publication if further clarification is required.

### 1.3.1 $H$-valued Lévy Process and its Stochastic Calculus

In Chapter 2, we review the basic properties of Lévy processes in Hilbert space. We begin with a few well-known theorems describing the structure of Lévy processes. Then we study stochastic integrals w.r.t. Hilbert space-valued Lévy processes not necessarily square integrable.

In section 2.3, we present a new version of stochastic Fubini theorem for Lévy process, which is developed in attempting to recover the Da Prato-Kwapień-Zabczyk's factorization technique which is used for $p > 2$ moments. The precise result we prove is the following.

**Theorem 3.3.4.** *Let $U$ and $E$ be separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $T > 0$ be fixed. Assume that the mapping $[0, T] \times [0, T] \times \Omega \ni (s, \sigma, \omega) \mapsto \Phi(s, \sigma, \omega) \in L(U, E)$ is a strongly measurable with respect to the $\sigma$-algebra $\mathcal{B}([0, T]) \otimes \mathcal{P}_T$, where $\mathcal{P}_T$ stands for the predictable $\sigma$-algebra in $[0, T] \times \Omega$. More precisely, we assume that for every $y \in U$ the mapping $[0, T] \times [0, T] \times \Omega \ni (s, \sigma, \omega) \mapsto \Phi(s, \sigma, \omega)y \in E$ is measurable with respect to the $\sigma$-algebra $\mathcal{B}([0, T]) \otimes \mathcal{P}_T$. Furthermore, assume that $L$ is a $U$-valued Lévy process defined as $L(t) := W(Z(t))$, $Z(t)$ is a subordinator process belonging to Sub(p), i.e. $Z(t)$ has intensity measure satisfying*

$$\rho(\{0\}) = 0, \quad \int_1^\infty \rho(d\xi) + \int_0^1 \xi\rho(d\xi) < \infty \quad \int_0^1 \xi^{\frac{p}{2}}\rho(d\xi) < \infty$$

*where $\rho$ and $\nu$ are respectively the intensity measure on $\mathbb{R}$ and Lévy measure on $U_0$. One relates $\rho$ and $\nu$ as*

$$\nu(\Gamma) = \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \Gamma \in \mathcal{B}(Y),$$

*Then*

$$\int_0^T \left( \int_0^s \Phi(s, \sigma) dL(\sigma) \right) ds = \int_0^T \left( \int_\sigma^T \Phi(s, \sigma) ds \right) dL(\sigma)$$

One must also point out that we were not able to recover the Da Prato-Kwapień-Zabczyk's factorization technique, due to the contradiction between the fractional parameter of the Riemann Liouville operator and the stability index of stable process. Nevertheless, the stochastic Fubini theorem is quite new compare to other versions in the literature which were developed for stochastic integrals with respect to the compensated Poisson random measure in martingale type spaces [118]. One must point out that there are different versions of stochastic Fubini theorems that have been studied by many authors, see the books [37, 87] and also the PhD thesis [118]. However, these versions do not cover the interesting stable process case which we do.
In section 2.4, we review the theory of subordinators and $H$-valued subordinated processes, based on the publication [23], and complete some missing proofs of several important results.
In view of the above, the main contribution of Chapter 3 is

- A new stochastic Fubini theorem (Theorem 2.4.4) covering a broad class of functions with respect to Poisson random measure.

*Notation* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space in which an increasing and right continuous family of complete $\sigma$-algebras $\{\mathcal{F}_t, t \geq 0\}$ of $\mathcal{F}$ is defined. Let $u(t,x)$, $\nu$ and $p(t,x)$ be respectively the velocity, viscosity parameter and pressure of an incompressible fluid. Moreover, Let $W_Q$ be a $Q$-Wiener process. The distributional derivative of $W_Q(t)$, $t \geq 0$ represents one source of external force acting on the fluid. When $Q = I$, we have the cylindrical Wiener process. Let $(\cdot, \cdot)$ denotes the inner product in Hilbert space and denote the induced duality by $\langle \cdot, \cdot \rangle$. The space $D([0,T]; H)$ is the space of all right-continuous with left-hand limits functions endowed with the usual Skorohod topology maps from $[0,T]$ to $H$. The element of $D([0,T]; H)$ is defined on $[0,T]$ and takes value in $H$.

### 1.3.2 Well-posedness

It is well accepted fact that SPDE do not possess exact solutions, but rather probability distribution. Hence, the first question came into our mind is: in what sense can we solve SNSE with stable Lévy noise. In 2D, as mentioned earlier, it is now well known that the SNSE with both Gaussian and certain Lévy noise has unique global strong solution [40, 50, 78]. The fundamental idea for establishing well-posedness is to find conditions to ensure existence and uniqueness of solutions. In the first part of Chapter 4, we will prove the global existence (and uniqueness) of a weak and strong solution of (1.2). Let us now summarise the key well-posedness results obtained in this thesis.

#### 1.3.2.1 Weak solutions

It is a well known fact in the theory of SPDE (or PDE) that, many equations do not have classical solutions and have to be solved in some weaker sense. Heuristically speaking, when seeking for solutions, one often starts with finding the so-called weak solution. Let us consider the SNSE as an abstract Itô equation in variational form

$$d(u(t), v) + \langle \langle Au(t) + B(u(t)) + Cu(t), v \rangle dt = (f, v)dt + (GdL(t), v)$$

with initial condition

$$(u(0), v) = (u_0, v)$$

for any $v$ in the space $V$. This requires the following assumptions on the initial data

$$f \in V', \quad z \in L^4_{\text{loc}}([0, \infty); \mathbb{L}^4 \cap H), \quad u_0 \in H$$

and the solution is expected to be an adapted (and measurable) stochastic process $u = u(t, x, \omega)$ satisfying

$$u \in D(0, T; H) \cap L^2(0, T; V)$$

In Chapter 3 we study existence and uniqueness of solutions to (1.2) and the existence of an invariant measure. We prove six new theorems, in which we gain a complete understanding of the well-posedness of (1.2). One must point out that our proof of existence and uniqueness of a weak solution (See Theorem 3.2.11 ) is inspired from the Gaussian counterpart which has been thoroughly investigated in Goldys et al. [14].

In Theorem 3.2.11, we prove the existence and uniqueness of a weak solution (or the pathwise variational solution) to the SNSE on the rotating sphere for a $H$-valued Cylindrical stable noise. Theorem 3.2.11 states that,

For any $\alpha \geq 0$, $z \in L^4_{\text{loc}}([0,\infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$ and $f \in V'$, there exists a unique solution $v$ of equation (3.75).

Let us point out here that Theorem 3.2.11 mimics Theorem 3.2 in Gaussian case [14], given the suitable conditions of noise are assumed. Here we sketch some main lines of the proof of 3.2.11. Using the Leray-Helmholtz projection, we recast (1.2) into a functional equation in $H = L^2(\mathbb{S}^2)$:

$$du(t) + Au(t)dt + B(u(t), u(t))dt + Cu = fdt + GdL(t), \quad u(0) = u_0.$$

Then for any $\alpha > 0$, by introducing a stationary Ornstein Uhlenbeck process $z_\alpha$ which satisfies the linear equation (the auxiliary Ornstein Uhlenbeck process)

$$dz_\alpha + (\nu A + C + \alpha)z_\alpha dt = GdL(t), \quad t \in \mathbb{R},$$

where $G : H \to H$ is a bounded linear operator, $L(t)$, $t \in \mathbb{R}$ is a two-sided Lévy process as defined in (1.2). Let us remark that $\alpha > 0$ is arbitrary in the proof of existence and uniqueness but one would have to choose cautiously for the proof of existence of random attractors. As we will show in section 3.3 the process $v(t) = u(t) - z_\alpha(t)$ is a solution to the deterministic nonlinear PDE for each fixed trajectory of the process $z_\alpha$

$$\begin{cases} \frac{d^+}{dt}v(t) + (\nu A + C)v(t) = f - B(v(t) + z_\alpha(t)) + \alpha z_\alpha(t) \\ v(0) = v_0 \end{cases} \tag{1.7}$$

where $\frac{d^+}{dt}$ denotes the right hand derivative at $t$. From this point onward, the existence and uniqueness of (1.2) reduces to the existence and uniqueness of solutions to the classical Navier-Stokes equations on the sphere with Coriolis force. This question has been extensively studied as mentioned earlier. See, for instance, Il'in and Filatov [65]. Thereby one can still use the classical Galerkin approximation based on expansion into series of vector spherical harmonics. Thanks to theorem 8.1 in [23], the solution to the auxiliary Ornstein Uhlenbeck (OU) equations has a $L^4$-solution. Hence, based on this hypothesis, the proof of existence and uniqueness of (1.2), continuous dependence on initial data, forcing and driving noise follows the same lines as in Goldys et al.[14], which was achieved in a similar fashion as in the 2D bounded domain case. This is because, the equations are considered on the surface of the sphere, and the presence of the Coriolis force leads to minor modifications in the proof. Let us review the proof in [14]. The existence of weak solutions is proved via the Galerkin approximation argument. The basic idea that underpins this method is to approximate $v : [0, T] \to H$ by functions $v_L : [0, T] \to P_L$ that take values in a finite-dimensional subspace $H_L \subset H$ of dimension $L$. More precisely, the existence of weak solutions is proved in three steps [47]:

- Construction of approximate solutions;
- Derivation of energy estimates for approximation solutions;
- Convergence of approximate solutions to a solution;

*Step 1* To construct an approximation, for any $L \in \mathbb{N}$, take

$$H_L = \text{linspan}\{\mathbf{Z}_{l,m} : l = 1, \cdots, L; |m| \leq l\}$$

as the linear space spanned by the first $L$ eigenfunctions in an orthonormal basis $\{\mathbf{Z}_{l,m} : l = 1, \cdots, L; |m| \leq l\}$ of $H$, which may be assumed to be orthogonal in $V$. In other words, $H_L$ is an $L$-dimensional subspace of $V$ and $P_L$ is the orthogonal projection from $H$ onto $H_L$ defined as

$$P_L v = \sum_{l=1}^{L} \sum_{m=-l}^{l} (v, \mathbf{Z}_{l,m}) \mathbf{Z}_{l,m}.$$

One projects the evolution equation (1.7) onto $L^2([0, T]; H_L)$ and considers the following approximate problem for (1.7) on the finite dimensional space $H_L$ (From now on we will write $\partial_t$ for $\frac{d^+}{dt}$):

$$\begin{cases} \partial_t v_L(t) = P_L[-\nu A v_L - B(v) - B(v, z) - B(z, v) - \mathbf{C} v(t) + F], \\ v(0) = P_L v_0, \end{cases} \tag{1.8}$$

where $F = -B(z) + \alpha z + f$ and operators $A$, $B$ and $\mathbf{C}$ are defined in (1.2). The solution $v_L$ is said to be the Galerkin approximation. In this way, $v_L$ is required to satisfy (1.7) up to a residual which is orthogonal to $H_L$. Notice that the system (1.8) is essentially a system of ODE for $v_L$ which has the equivalent form

$$\frac{d^+ v_L}{dt} = G(t, v_L(t)), \quad t \geq 0,$$

where the function $G(t, v)$ is an $H_L$-valued locally Lipschitz continuous with respect to $v \in H_L$ and measurable with respect to $t$. It follows from the theory of ODE that the above system of nonlinear differential equations has a unique solution defined on some maximum time interval $[0, T_L)$.

*Step 2* Extend the solution $v_L$ globally in time. Toward this end, one derives some energy estimates in form of uniform a priori estimates (uniformly in $L$) and check if the solution is bounded for all time. Take the inner product of (1.8) in $H$ with $v_L(t)$ one obtains

$$(\partial_t v_L(t), v_L(t)) = -\nu(P_L A v_L, v_L) - (P_L B(v_L), v_L) - (P_L B(v_L, z), v_L) - (P_L B(z, v_L), v_L) - (P_L \mathbf{C} v_L, v_L) + \langle F, v_L\rangle$$

Using the identities

$$(\partial_t v_L(t), v_L(t)) = \frac{1}{2}\frac{d^+}{dt}|v_L(t)|^2,$$
$$-\nu(P_L A v_L, v_L) = -\nu(A v_L, v_L) = -\nu|v_L|_V^2,$$

$$(P_L B(v_L), v_L) = (B(v_L), v_L) = b(v_L, v_L, v_L) = 0, \quad (P_L B(z, v_L), v_L) = b(z, v_L, v_L) = 0,$$

$$(P_L \mathbf{C} v_L, v_L) = (\mathbf{C} v_L, v_L) = 0.$$

One obtains for all $t > 0$

$$\frac{1}{2}\frac{d^+}{dt}|v_L(t)|^2 = -\nu|v_L(t)|_V^2 - b(v_L(t), v_L(t), v_L(t)) + \langle F(t), v_L(t)\rangle \quad t \in [0, T_L).$$

Now one estimates the term $\frac{1}{2}\frac{d^+}{dt}|v_L(t)|^2$ by estimating the right hand side of the above equality. Using a basic trilinear estimate combine with the Young inequality one has that,

$$|b(v_L, v_L, z)| \leq \frac{C}{\nu^3}|v_L|^2|z|_V^4 + \frac{\nu}{4}|v_L|_V^2,$$

and

$$\langle F(t), v_L\rangle \leq |F(t)|_{V'}|v_L|_V \leq \frac{1}{\nu}|F(t)|_{V'}^2 + \frac{\nu}{4}|v_L|_V^2.$$

Therefore one obtains

$$\partial_t|v_L(t)|^2 + \nu|v_L|_V^2 \leq \frac{C}{\nu^3}|v_L|^2|z|_V^4 + \frac{2}{\nu}|F(t)|_{V'}^2, \quad t \in [0, T_L). \tag{1.9}$$

Invoking Gronwall Lemma, one has

$$|v_L(t)|^2 \leq |v(0)|^2 \exp\left(\frac{C}{\nu^3}\int_0^t |z(\tau)|_V^4 d\tau\right) + \int_0^t \frac{2}{\nu}|F(s)|_{V'}^2 \exp\left(\frac{C}{\nu^3}\int_s^t |z(\tau)|_V^4 d\tau\right) ds, \quad t \in [0, T_L).$$

Integrating (1.9) in time from 0 to $T$ and using the above inequality we obtain

$$|v_L(T)|^2 + \nu\int_0^T |v_L(t)|_V^2 dt + \frac{C}{\nu^3}\int_0^T |z(t)|_V^2|v_L(t)|^2 dt + \frac{2}{\nu}\int_0^T |F(t)|_{V'}^2 dt.$$

In view of the above it is shown that $v_L$ is uniformly bounded (in $L$ ) in the norm $L^\infty(0, \infty; H)$ and $L^2(0, \infty; V)$.

In order to pass to the limit (See Step 3), one also has to show that $v_L$ converges to $v$ in the strong topology $L^2(0, T; H)$. For this one needs an a priori estimate on a fractional derivative in time of the approximate solution. By taking Fourier transforms (in the time variable $t$), one proves that

$$\text{the sequence } \{\tilde{v}_L, L \in \mathbb{N}\} \text{ is bounded in } H^\gamma(0,T;\mathbb{H}^1(\mathbb{S}^2),\mathbb{L}^2(\mathbb{S}^2)). \tag{1.10}$$

where the space $\mathcal{H}^\gamma$ is defined as

$$\mathcal{H}^\gamma(\mathbb{R};X_0,X_1) = \{v \in L^2(\mathbb{R};X_0) : D_t^\gamma v \in L^2(\mathbb{R};X_1)\} \quad \gamma > 0.$$

*Step 3* Here one uses a compactness argument prove that a subsequence of approximate solutions converges to a weak solution to (1.7). In this case, using the Banach-Alaogu theorem, the existence of uniform bounds in $L^\infty(0,\infty;H)$ and $L^2(0,T;V)$ allows one to assert the existence of an element $v \in L^2(0,T;V) \cap L^\infty(0,\infty;H)$ such that

$$\begin{cases} v_L \rightharpoonup v, & \text{weakly in } L^2(0,T;V), \\ v_L \rightharpoonup v, & \text{weakly}^* \text{ in } L^\infty(0,T;H) \quad \text{as } L \to \infty. \end{cases} \tag{1.11}$$

Now one needs the following compactness theorem (for fractional derivative ).

**Theorem 1.3.1** (Chapter III, Theorem 2.2 [105]). *Suppose that $X_0 \subset X \subset X_1$ is a Gelfand triple of Hilbert spaces and the injection of $X_0$ into $X$ is compact. Then for any bounded set $K \subset \mathbb{R}$ and $\gamma > 0$, the injection of $\mathcal{H}_K^\gamma(\mathbb{R};X_0,X_1)$ into $L^2(\mathbb{R};X)$ is compact.*
Using this result and the estimate (1.10), we deduce that

$$v_L \to v \quad \text{strong in } L^2(0,T;\mathbb{L}^2(\mathbb{S}^2)). \tag{1.12}$$

The convergence results (1.11) and (1.12) allow one to pass to the limit. Now one claims that the weak limit $v$ is indeed a weak solution to the evolution equation (1.7). Toward this end, it suffices to check that for any differentiable function $\psi : [0,T] \to \mathbb{R}$ such that $\psi(T) = 0$ and $\phi \in H_l \subset H_l$ for some $l \in N^+$ the following approximate equation

$$\begin{aligned} -\int_0^T (v_L(t), \psi'(t)\phi)dt = & -\nu \int_0^T (P_L A v_L(t), \psi(t)\phi)dt \\ & - \int_0^T (P_L B(v_L(t)), \psi(t)\phi)dt - \int_0^T P_L B(v_L(t), z, \psi(t)\phi)dt \\ & - \int_0^T P_L B(z, v_L(t), \psi(t)\phi)dt - \int_0^T \langle P_L F(t), \psi(t)\phi\rangle dt + (v_L(0), \psi(0)\phi) \end{aligned} \tag{1.13}$$

converges to

$$
\begin{aligned}
-\int_0^T (v(t), \psi'(t)\phi)dt = -\nu \int_0^T (v(t), \psi(t)\phi)dt \\
- \int_0^T b(v(t), v(t), \psi(t)\phi)dt - \int_0^T B(v, z, \psi(t)\phi)dt \\
- \int_0^T B(z, v(t), \psi(t)\phi)dt - \int_0^T \langle F, \psi(t)\phi\rangle dt + (v(0), \psi(0)\phi) \qquad (1.14)
\end{aligned}
$$

Passing to the limit with the sequence $L$ is easy for the linear terms in (1.13): $-\nu \int_0^T (P_L A v_L(t), \psi(t)\phi)dt$ and $-\int_0^T \langle P_L F(t), \psi(t)\phi\rangle dt + (v_L(0), \psi(0)\phi)$, since the arguments under the integral are clearly bounded. The other three nonlinear terms can also be shown to be bounded using the following Lemma.

**Lemma** (Lemma 4.3[14]). *Suppose $u : [0, T] \times \mathbb{S}^2 \to \mathbb{R}^2$ is a $C^1$ function and all first derivatives of components of $u$ are bounded on $\mathbb{S}^2 \times [0, T]$. Suppose $v_m \to v$ weakly in $L^2(0, T; V)$ and strongly in $L^2(0, T; \mathbb{L}^2(\mathbb{S}^2))$. Then*

$$
\int_0^T b(v_m(t), v_m(t), u(t))dt \to \int_0^T b(v(t), v(t), u(t))dt
$$

In the same situation as in other similar compactness arguments, one has existence but not uniqueness since the subsequences of approximate solutions $v_L$ could converge to a different weak solutions. Nevertheless, the uniqueness of the solution to (1.7) can be established using the classical argument of Lion and Prodi [74] with slight modification due to the additional noise term. Using a similar argument to the proof of uniqueness, the continuous dependence on the deterministic force and the driving noise is also established.

#### 1.3.2.2 Strong solutions

By imposing higher regularity on the forcing term, we then prove a new existence theorem for the strong solution

**Theorem 3.2.16.** *If*

$$
z \in L^4_{loc}([0, \infty); \mathbb{L}^4(\mathbb{S}^2) \cap H), \quad f \in H, \ v_0 \in H,
$$

*then $u(t) \in V$ for all $t > 0$ almost surely. Moreover, if*

$$
\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty \,,
$$

*then the theorem holds.*

Our methodology to prove Theorem 3.2.16 is motivated by a fixed point argument used in Brzeniak, Capinski and Flandoli [19] in their study of the existence of random dynamical system (stochastic flow) $\varphi$. The authors were mainly interested in the existence of global (random) attractors. In more detail, the authors in [19] study the following 2D abstract Navier-Stokes equations with real noise in a Hilbert space:

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = B(u(t), \xi(t)) + f, & t \geq 0 \\ u(0) = u_0, \end{cases} \tag{1.15}$$

where $\xi(t)$ is a continuous stationary process on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $E$ and is defined as $\xi(t,\omega) := \omega(t)$, $\omega \in \Omega$. Here, $E$ is a closed non-empty subset of $\mathbb{R}^m$ and $\Omega = D_E([0,\infty))$ is the space of all RCLL functions endowed with the usual skorohod topology (See Chapter 4 for definition.) The mild form of (1.15) is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)(B(u(s), \xi(s)) + f)ds. \tag{1.16}$$

They prove the following theorem, via a fixed point argument, which contains two important results: existence and uniqueness of a strong solution and the existence of a stochastic flow $\varphi$.

**Theorem 3.1** [19]. *For all $u_0, f \in H$ and $\omega \in \Omega$ there exists a unique solution $u \in C(0,\infty; H) \cap L^2_{loc}(0,\infty; V)$ of the mild solution (1.16), which belongs to $C(\varepsilon,\infty; V) \cap L^2_{loc}(\varepsilon,\infty; D(A))$ for all $\varepsilon > 0$. Moreover, if $u_0 \in V$, then $u \in C(0,\infty; V) \cap L^2_{loc}(0,\infty; D(A))$. Finally, if we define $\varphi : T \times \Omega \times H \to H$ as*

$$\varphi_t(\omega)u_0 = u(t;\omega;u_0), \quad t \in T, \omega \in \Omega, u_0 \in H$$

*where $u(\cdot;\omega,u_0)$ is the solution to (1.16) corresponding to given $\omega \in \Omega$ and $u_0 \in H$, then $\varphi$ satisfies the assumptions (1)-(4).*

The proof of Theorem 3.1 in Brzeniak, Capinski and Flandoli [19] follows three main steps. The *first* step is to establish local existence and uniqueness of a solution of (1.16) with initial data $u_0 \in V$, using the classical contraction mapping theorem. One proves that a strong solution at least exists locally in time whenever the initial condition $u_0 \in V$. *Next*, one constructs a global solution with initial data $u_0 \in V$. This is achieved by proving a *a priori estimate* in $V$ with initial condition $u_0 \in V$. The *Last* step is to construct a global solution with initial data $u_0 \in H$. This is achieved by approximation. Using the fact $V$ is dense in $L^2$, one takes a sequence of solutions $\{u_n\}$ of (1.16) in the space $Y_T := C(0,T;V) \cap L^2(0,T;D(A))$ with initial data $\{u_{0,n}\} \subset V$ converging to $u_0$ in $H$, and $u_n$ can be shown to be Cauchy. Moreover, the limit

function is shown to satisfy (1.16). This concludes the claim of the global existence of strong solution. Under this fixed point argument, we are able to prove the existence (and uniqueness) of strong solution in our case. Following the same strategy in the weak solution case, we make change of variable $v(t) = u(t) - z(t)$, prove Theorem 3.2.16 by first studying the well-posedness of a strong solution to the transformed problem. In section 3.4, using the usual contraction mapping principle, we prove a strong solution exists at least locally in time whenever the initial condition $u_0$ belongs to space $V$. Then following the same steps as described above in [19] we show the solution exists globally in time via a successive approximation of contractions.
Finally, Let us point out that our proof of the existence theorem (Theorem 3.2.16) can also be accomplished under the classical Galerkin approximation argument.

### 1.3.3 Invariant Measures

Invariant measures of stochastic Navier-Stokes equations with Lévy noise has been studied in the paper [11, 41]. In the final section of chapter 3, we prove the existence of invariant measure of (1.2) by establishing the three usual criteria: Markov property, Feller property, and tightness of the probability law. The first two properties follow naturally from our study of well-posedness in Chapter 3 (see subsections 3.5.1). Tightness is given by the Krylov-Bogoliubov argument for Markov processes, which requires some a priori estimates. In obtaining these a priori estimates, we initially attempted to mimic the arguments in Chapter 15.4 of the book [36]. A major difficulty arised when analyzing the estimate $\frac{d^+}{dt}|v(t)|^2$: the usual estimates for the nonlinear term $b(v(t), z(t), v(t))$ yield a term $|v(t)|^2|z(t)|_4^4$, so when averaging over time, we were not able to deduce any bound in $H$ for $|v(t)|^2$ under classical lines as the Birkhoff ergodic theorem fails. Let us now analyse the argument in [36] and see why this classical method fails to be applicable, that is, when the noise process does not have a finite second moment (our case).
Flandoli [52] considers the following initial-boundary value problem where the stochastic Navier-Stokes equation has additive Gaussian white noise in 2D bounded domain $D \subseteq \mathbb{R}^2$:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + (u \cdot \nabla)u + \nabla p + dW_Q(t), & \text{on } [0,\infty) \times D, \\ u(t,x) = 0, & \text{on } [0,\infty) \times \partial D, \\ \operatorname{div} u(t,x) = 0, & \text{on } [0,\infty) \times D, \\ u(0,x) = u_0(x) \quad on D. \end{cases}$$

*Notation*: Let $H$ be the closure of the set $\{u \in C_0^\infty(D, \mathbb{R}^2) : \nabla \cdot u = 0\}$ in the $L^2$-norm, $|u| = (u,u)^{1/2}$ and $(u,v) = \sum_{j=0}^{2} u^j(x) v^j(x) dx$.

Let $V$ be the closure of the set $\{u \in C_0^\infty(D, \mathbb{R}^2) : \nabla \cdot u = 0\}$ in the norm $|u| + |u|_V$ where

$$|u|_V^2 = \int_D (|D_x u(x)|) dx.$$

More generally, for any $p \geq 2$, with slight abuse of notation, denote by $L^p$ the closure of the set $\{u \in C_0^\infty(D, \mathbb{R}^2) : \nabla \cdot u = 0\}$ with respect to the norm

$$|u|_p = \left( \int_D |u(x)|^p dx \right)^{1/p}.$$

Using the Leray-Helmholz projection, the equation is reformulated in a standard way as an abstract stochastic evolution equation

$$du(t) = Au(t)dt + B(u(t), u(t))dt + dW_Q(t) \tag{1.17}$$

in a Hilbert space $H$ which is the closure of the space of compactly supported solenoidal $C^\infty$-smooth 2D vector fields on $D$ in the $[L^2(D)]^2$ topology. The process $W$ is defined as

$$W(t) = \sum_{l=1}^{\infty} \sigma_l \beta_l g_l$$

where $\{\beta_l\}$ is a sequence of mutually independent real Brownian motions defined on a filtered probability space. Moreover, $\{\sigma_l\}$ is a sequence of positive numbers and $\{g_l\}$ is an orthonormal basis in $H$. Following the general way to construct solutions in the additive noise case, set

$$v = u - W_A(t).$$

Then (1.17) becomes,

$$\begin{cases} \frac{dv}{dt} = Av(t) + D_x(v(t) + W_A(t))(v(t) + W_A(t))dt + f(t), \quad t \geq 0, \quad v(0) = x \end{cases} \tag{1.18}$$

where

$$W_A(t) = \int_0^t S(t-s)dW(s).$$

The local existence and uniqueness of the solution to the stochastic Navier-Stokes equations is obtained via a simple fixed point method based on the assumption that the stochastic convolution $W_A(t)$ is $L^4$ trajectories integrable, i.e.

$$\left( \int_0^T |W_A(t)|_4^4 \right) < \infty.$$

Then via the a priori estimates in $H$ and $V$, the solution is shown to exist globally in time.
A third useful a priori estimate, which plays an important role in the proof of existence of invariant measure is the following

$$\frac{d}{dt}|v(t)|^2 + |v(t)|_V^2 \leq (\varepsilon + K_1|\varphi(t)|_4^4)|v(t)|^2 + K_2|\varphi(t)|_4^4 + \frac{4\alpha^2}{\varepsilon^2}|\varphi(t)|^2, \quad t \geq 0, \tag{1.19}$$

where $\alpha > 0$, $\epsilon \geq 0$. $\varphi(t) = W_A(t,\omega)$ and $y$ is a mild solution of

$$\frac{dv}{dt} = Av(t) + D_x(v(t) + \varphi(t)) \cdot (v(t) + \varphi(t)) + \alpha\varphi(t)$$

With this estimate, together with the hypothesis that $W_A(\cdot)$ is continuous in $D((-A)^{\frac{1+2\theta}{4}})$ for some $\theta \in (0, 1/2)$, the existence of invariant measures is established in [36]:

**Theorem 15.4.1.** *Under the hypothesis $W_A(\cdot)$ is continuous in $D((-A)^{\frac{1+2\theta}{4}})$ for some $\theta \in (0, 1/2)$, there exists at least one invariant measure for equation (1.2).*

The proof of tightness relies on a dissipativity condition. Roughly speaking, one needs to find an almost surely finite real random variable $r(\omega)$ such that

$$\sup_{-\infty < t_0 \leq 0} |(-A)^{\theta} u_{t_0}(0,\omega)| \leq r(\omega).$$

It is a well known fact that $D(A^{\delta})$ can be compactly embedded into the space $H$. If $u_{t_0}$ is denoted as the solution of (1.17) satisfying $u_{t_0}(\eta) = 0$; and if one can show that the random variables $(u_{t_0}(0), -\infty < \lambda \leq 0)$ are bounded in probability in the space $D((-A)^{\theta})$ for a certain $\theta \in (0, 1/2)$, then one concludes immediately that the laws $\mathcal{L}(X(t,0))$, $t \geq 1$ are tight on $H$.
The proof of Theorem 15.4.1 is in the book [36]. Let us sketch briefly the main steps as we use the same argument for a similar result in Chapter 3. The proof can be established by verifying the following two claims.
**Claim 1**
There exists $\alpha > 0$ and a random variable $\xi$ such that $\mathbb{P}$ almost surely

$$|v_{\alpha}(t,s)| \leq \xi \quad \forall\, t \in [-1, 0] \quad \text{and all} \quad s \leq -1, \tag{1.20}$$

$$\int_{-1}^{0} |v_{\alpha}(t,s)|_V^2 ds < \xi \quad \forall\ t \in [-1, 0] \text{ and all } s \leq -1. \tag{1.21}$$

**Proof.** Let $(\tilde{W}(t), t \geq 0)$ be a Lévy process that is an independent copy of $W$. Denote by $\bar{W}$ a Lévy process on the whole real line by

$$\overline{W}(t) \begin{cases} W(t), & t \geq 0 \\ \tilde{W}(-t), & t < 0 \end{cases} \tag{1.22}$$

and by $\bar{\mathcal{F}}_t$ the filtration

$$\bar{\mathcal{F}}_t = \sigma(\overline{W}(s), s \leq t), \quad t \in \mathbb{R}.$$

First, Let $z_\alpha$ be the stationary solution of the equation

$$dz_\alpha = (A-\alpha)z_\alpha dt + dW(t).$$

Since $z$ is assumed to be $D((-A)^\theta)$ continuous, $z_\alpha$ is also $D((-A)^\theta)$ continuous as $z_\alpha$ relates to $z$ in the following manner,

$$z_\alpha = z(t) + e^{(A-\alpha)(t-s)}(z_\alpha - z(s)) - \alpha \int_s^t e^{(A-\alpha)(t-\sigma)} z(\sigma) d\sigma, \quad t \geq s.$$

Moreover, Let

$$v_\alpha(t,s) = u(t,s) - z_\alpha(t), \; t \geq s.$$

Then for any $t \geq s$, $v_\alpha(t) = v_\alpha(t,s)$ is the mild solution of the equation

$$\begin{aligned} \frac{d}{dt} v_\alpha(t) &= A v_\alpha(t) + D_x(v_\alpha(t) + z_\alpha(t)) \cdot (v_\alpha(t) + z_\alpha(t)) t \geq s, \\ v(s) &= -z_\alpha(s). \end{aligned}$$

Taking into account of the a priori estimate (1.19) and the classical estimate

$$|v|_{D((-A)^{1/2})} \geq \lambda_1 |v|^2$$

one has

$$|v_\alpha(t,s)|^2 \leq e^{\int_s^t [-\lambda_1+\epsilon+K_1|z|_4^4] d\sigma} |v_\alpha(s)|^2 + \int_s^t e^{\int_\sigma^t [-\lambda_1+\epsilon+K_1|z|_4^4] d\zeta} (K_2 |z_\alpha(\sigma)|^4 + \frac{4\alpha^2}{\epsilon^2} |z_\alpha(\sigma)|^2) d\sigma.$$

It is well known that the Ornstein-Uhlenbeck processes $z_\alpha$ has a unique invariant (and Gaussian) measure. So Let $\mu_\alpha$ be the unique invariant measure for $z_\alpha$ supported by $D((-A)^{\frac{1+2\theta}{4}}) \subset L^4(D)$, and therefore, by the strong law of large numbers, one has $\mathbb{P}$ almost surely that

$$\lim_{s \to -\infty} \int_s^t |z_\alpha(\sigma)|_4^4 d\sigma = \int_V |z|_4^4 \mu_\alpha(dz).$$

Moreover,

$$\lim_{\alpha \to +\infty} \int_V |z|_4^4 \mu_\alpha(dz) = 0.$$

Consequently one can find $\alpha > 0$ and $\epsilon > 0$ such that

$$-\lambda_1 + \epsilon + K_1 \int_V |z|_4^4 \mu_\alpha(dz) = -\gamma < 0. \tag{1.23}$$

□

**Remark.** In view of (1.23), the strong law of large number does not apply in our case, as the stable process does not even possess a finite second moment.

**Claim 2**
Assume that $z$ is a continuous and stationary Gaussian process on a separable Banach space $L$. Then for arbitrary $\delta > 0$, there exist random variables $\xi_1$ and $\xi_2$ such that $\mathbb{P}$-almost surely,

$$|z(t)|_L \leq \xi_1 + \xi_2 |t|^{\delta} \quad \forall t \leq 0.$$

**Proof.** The proof uses the Chebychev inequality and Borel-Cantelli Lemma; we refer readers to p.288-289 in [36]. □

The proof of Proposition 15.4.3 is completed once the proofs of Claim 1 and Claim 2 are established.
To overcome the difficulty demonstrated in the above Remark, we assume finite dimensional noise, which was inspired by an argument proposed by Flandoli and Crauel [34], who introduced it when studying global random attractors for the RDS generated by a SNSE with additive noise. Let us present our results on tightness. We consider

$$du(t) = [-Au(t) - B(u(t), u(t)) + \mathbf{C}u(t) + f]dt + \sum_{l=1}^{m} \sigma_l e_l dL_l(t) \tag{1.24}$$

where operators $A$, $B$, $C$ are defined in (1.2), $f \in H$, $L_1$, $L_2 \cdots L_l$ are i.i.d. $\mathbb{R}$-valued symmetric $\beta$-stable processes on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\sigma$ is a bounded sequence of real numbers and $e_l$ is the complete orthonormal system of eigenfunction on $H$.
We work based on the change of variable $v = u - z$ where $z(t) = \sum_{l=1}^{m} e_l z_l(t)$ is the solution to the Ornstein-Uhlenbeck equation

$$dz + (\hat{A} + \alpha I)zdt = \sum_{l=1}^{m} \sigma_l e_l L_l(t)$$

and $v$ satisfies

$$\frac{dv^+}{dt} = -\nu A v(t) - Cv(t) - B(u, u) + f + \alpha z.$$

A key estimate (which is analogous to (1.19)) is given as the following.

**Proposition 3.5.7.** *Let $\alpha > 0$, $v$ be a mild solution of (3.187), there exist constants $c, c' > 0$ depending only on $\lambda_1$ such that*

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \leq \left(-\frac{\lambda_1}{4} + 2\eta \sum_{l=1}^{m} |z_l|\right)|v|^2 + c|f|^2 + c\alpha|z|^2 + 2\eta|z|^2 \sum_{l=1}^{m} |z_l|. \tag{1.25}$$

The proof of Proposition 3.5.7 relies on the following nonlinear estimate, (which was motivated from Crauel and Flandoli [34]),

$$\begin{aligned}\langle B(u,z),u\rangle &= \sum_{l=1}^{m}\langle B(u,e_l),u\rangle z_l \le \eta|u|^2\sum_{l=1}^{m}|z_l|\\ &\le 2\eta|v|^2\sum_{l=1}^{m}|z_l| + 2\eta|z|^2\sum_{l=1}^{m}|z_l|\end{aligned}$$

and the fact $z(t)$ is an ergodic process which is supported by $D(A^\delta)\subset L^4(\mathbb{S}^2)$.
With the aid of the finite dimensional estimate of a nonlinear term and the ergodic property of $z(t)$ in Lévy case. We now can use the classical argument in Chapter 15 [36] to deduce the boundedness of the solution. Moreover, provided the noise does not degenerate (Lemma 4.5.9), the tightness of the probability law in $H$ is established.

**Proposition 3.5.8.** *There exists $\alpha > 0$ and a random variable $\xi$ such that $\mathbb{P}$-a.s.*

$$|v_\alpha(t,s)| \le \xi \quad \forall\, t\in[-1,0] \quad \text{and all} \quad s\le -1,$$
$$\int_{-1}^{0}|v_\alpha(t,s)|_V^2 ds < \xi \quad \forall\; t\in[-1,0] \text{ and all } s\le -1.$$

**Lemma 3.5.9.** *Assume that $X$ is a stationary process taking values in a Banach space $B$. Moreover, assume that for a certain $p > 0$ we have*

$$\mathbb{E}\sup_{t\in[-1,0]}|X(t)|_B^p < \infty\,.$$

*Then for every $\kappa > 0$ such that $\kappa p > 1$ there exists a random variable $\xi$ such that $\mathbb{P}$ a.s.*

$$|X(t)|_B \le \xi_1 + 2^\kappa|t|^\kappa$$

*for all $t\le 0$.*

Finally using the Chebyshev inquality we deduce that the family of probability measures lie in a compact set up to some small $\epsilon$. Hence the existence of invariant measures is established.

**Theorem.** *Assume additionally, that there exists $m > 1$ such that $\sigma_l = 0$ for all $l \ge m$. Then the solution $u$ to (3.61) admits at least one invariant measure.*

Moreover, we established the measure support in $D(A^\delta)$, $\delta\in[0,1/2]$.

**Proposition.** *For any $\delta \in [0, \frac{1}{2}]$, there exists $C = C(\delta)$ such that for any mild solution $v(\cdot)$ of (3.187), one has*

$$|A^{\delta}v(t)|^2 \leq e^{C\int_0^t |v(s)|^2|A^{\frac{1}{2}}v(s)|^2 ds}|A^{\delta}v(0)|^2 + C\int_0^t e^{C\int_\sigma^t |v(s)|^2|A^{\frac{1}{2}}v(s)|^2 ds}(|A^{\delta+\frac{1}{2}}f|^2 + |z(\sigma)|^2 + |A^{\frac{1+2\delta}{4}}z(\sigma)|^4)d\sigma. \tag{1.26}$$

#### 1.3.3.1 Contributions

In view of the above, the main contributions of Chapter 4 are

- A new theorem on the existence, uniqueness and continuous dependence on initial data for a pathwise variational weak solutions to the SNSE on the rotating sphere forced by cylindrical stable noise. Our results (See Theorems 3.2.11, 3.2.12, 3.2.13, 3.2.14) can be viewed as a generalisation of the existence and uniqueness results (Theorem 3.2 and 3.3) in Gaussian case in [14];
- Existence and uniqueness of a strong solution to the SNSE on the rotating sphere forced by cylindrical stable noise; (See Theorem 3.2.16)
- Existence of Invariant Measures to the SNSE on the rotating sphere forced by cylindrical stable noise.

### 1.3.4 Random Dynamical Systems

There are two types of attractors in teh theory of SDE (or SPDE) are worth distinguishable. One is the *measure attractor* w.r.t. the Markov semigroup generated by a SDE (SPDE). The second kind is the so-called random attractors which meant to be understood w.r.t. each trajectory of the random equation. To our awareness, the notion of measure attractor first appears in the paper [96], while random attractor were first defined in Crauel and Flandoli [34, 54]. The two notions of attractors were further unified in the paper [97].
The studies of attractors for random dynamical systems are In Chapter 4, we study the asymptotic behaviour of the functional NSE with force by an additive cylindrical stable noise

$$du(t) = [-Au(t) - B(u(t), u(t)) + \mathbf{C}u(t) + f]dt + \sum_{l=1}^{m} \sigma_l e_l dL_l(t), \tag{1.27}$$

where $L_l(t)$, $(1 \leq l \leq m)$ are mutually independent two-sided Lévy processes, $u = u(t, x, \omega)$ is now a random velocity of the fluid, and $e_l$ $(1 \leq l \leq m)$ the corresponding eigenfunctions of the stokes operator $A$ form an orthonormal basis in $H$. It is well known that the theory of random dynamical systems (RDS) permits one to study the qualitative behaviour of stochastic systems that are driven by many kinds of noise: from white noise, Markov processes, semimartingales to non-Markov process or fractional Brownian motion [45]. This motivates us to study the random dynamical system defined by the SNSE.

#### 1.3.4.1 Concepts of RDS

In section 4.2 we recall several key concepts in the theory of continuous RDS and convert to the jump case, such as random dynamical systems, double sided filtration. In section 4.2.4 we recall the concepts of invariant measures in RDS.
The first paper on random attractors is due to Brzezniak and Capinski [19]. The authors investigate the asymtotic behaviour of a general random dynamical system in a Polish space $X$ with some certain Borel $\sigma$-field. The Random Dynamical System $\varphi : \mathbb{R}^+ \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ is defined over $\vartheta$:

$$\varphi(t + s, \omega)x = \varphi(t, \vartheta_s \omega)\varphi(s, \omega)x \quad \forall \quad t, s \in \mathbb{R}^+, x \in X$$

$\mathbb{P}$ a.s. if and only

$$\varphi(t) \quad \text{is strongly measurable for all } t \in \mathbb{R}^+, \tag{1.28}$$

$$\varphi(t, \omega) : X \to X \quad \text{is continuous for all } t \in \mathbb{R}^+, \quad \mathbb{P} \text{ a.s.} \tag{1.29}$$

The trajectories $\varphi(t, \omega)x$ are right-continuous with left limits for every $x \in X, \mathbb{P}$ a.s.

It turns out here that the above definition of càdlàg RDS can be applied to this thesis (See Chapter 4.2.1). The difference in our case is that the time set $\mathbb{T}$ is chosen to be the whole of $\mathbb{R}$.
The notion of global attractors has drawn much attention in the theory of infinite dimensional dynamical systems over the last 50 years. This provides new insights into turbulence, see for instance Constantin and Foias [31], Constantin, Foias and Temam [30]. The basic framework of random dynamical systems (RDS) was developed mainly by Crauel, Debussche and Flandoli [33, 34]. See the monograph Arnold [7] for a comprehensive survey of RDS theory. The generalisation of this theory to the stochastic case has just been well developed in the last decade. The notion of a random attractor was first introduced by the earlier mentioned publication [19] to study random PDE of the form (1.15). It was later generalised to the context of stochastic PDE by Crauel and Flandoli [34]. The theory of random attractors plays an important role in the study of asymtotic behaviour of infinite dimensional dynamical systems. In contrast to the earlier work [19], Crauel and Flandoli [34] developed the basic theory of global random attractors for random dynamical systems. The random attractor is introduced as a random invariant set which is the $\Omega$-limit set at time $t = 0$ of the trajectories "starting in

bounded sets at time $t = -\infty$". Symbolically, the *random $\Omega$ limit set of a bounded set $B \subset X$ at time $t$* is defined as

$$A(t, \omega) = \overline{\cup_{B \subset X} A(B, t, \omega)}$$

and is called the random attractor. This notion of an attractor applies to RDS generated by evolution systems perturbed by white noise, in which case the classical notion of an attractor in deterministic case does not yield meaningful results. In this thesis, we adopt the notion introduced in Crauel and Flandoli [34] to define the abstract object random attractor, the existence of which is generated by the following result

**Theorem 5.3.9.** *Let $\varphi$ be an continuous in space, but càdlàg in time RDS on $X$ and assume the existence of a compact random set $K$ absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor $A$ given by*

$$A(\omega) = \overline{\bigcup_{B \subseteq X, B \text{ bounded}} \Omega_B(\omega)}, \quad \omega \in \Omega.$$

#### 1.3.4.2 Existence of stochastic flow $\varphi$

In section 4.3 we study the random dynamics generated by the SNSE, which is the core part of this chapter. In subsubsection 4.3.1, we construct the probility space for which the coordinate process $L_t(\omega, l) = \omega(l_t)$ is a two-sided Lévy process of $\beta$-stable type, $1 < \beta < 2$. In subsubsection 4.3.2 using property of analytic semigroups, and the fact $L_t(\omega) = \tilde{\omega}(t)$ has stationary increment, we show the flow $\hat{z}(t)$ is a.s. well defined. Moreover, using boundedness of noise, we show the map $\hat{z}$ is well defined, linear and bounded. In subsection 4.3.3, we prove that the SNSE (1.27) generates a RDS (see Theorem 4.3.5).
To prove our SNSE indeed defines a RDS, we proceed with three steps.
*Step 1* Identify a suitable canonical sample probability space for

$$du(t) = [-Au(t) - B(u(t), u(t)) + \mathbf{C}u(t) + f]dt + \sum_{l=1}^{m} \sigma_l e_l dL_l(t).$$

First, the state space is chosen as $E$, which is the completion of $A^{-\delta}(X)$ with respect to the image norm

$$|v|_E = |A^{-\delta} v|_X, \quad v \in X$$

where $\mathbb{L}^4(\mathbb{S}^2)$ is the space of Lebesgue measurable and fourth integrable vector fields on $\mathbb{S}^2$ and $X = \mathbb{L}^4(\mathbb{S}^2) \cap H$. The canonical sample space is chosen to be $\Omega = D(\mathbb{R}, E)$ of càdlàg functions, that is, functions that are right-continuous with left limits, defined on $\mathbb{R}$ and taking values in $E$.

**Remark.** The space $D(\mathbb{R}, E)$ is called the Skorohod space. It cannot be separable under the commonly used compact-open metric, that is,

$$\rho(\omega_1, \omega_2) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\omega_1(t) - \omega_2(t)|}{1 + |\omega_1(t) - \omega_2(t)|}, \quad |\omega_1(t) - \omega_2(t)|_n := \sup_{-n \le t \le n} |\omega_1(t) - \omega_2(t)|$$

cannot make $D(\mathbb{R}, E)$ separable. However, it can be made complete and separable when endowed with Skorohod metric [12] defined as the following

$$d(l_1, l_2) = \sum_{i=1}^{\infty} (1 \wedge d_i^\circ(l_1, l_2)) \quad \forall\ l_1, l_2 \in D$$

where $l_1^i(t) := g_i(t) l_1(t)$ and $l_2^i(t) := g_i(t) l_2(t)$ with

$$g_i(t) := \begin{cases} 1, & \text{if } |t| \le i - 1 \\ i - t, & \text{if } i - 1 \le |t| \le i \\ 0, & \text{if } |t| \le i \end{cases}$$

$$d_i^\circ(l_1^i, l_2^i) = \inf_{\lambda \in \Lambda} \left( \sup_{-i \le s < t \le i} |\log \frac{\lambda(t) - \lambda(s)}{t - s}| \vee \sup_{-i \le t \le i} |l_1(t) - l_2(\lambda(t))| \right),$$

where $\Lambda$ denotes the set of strictly increasing, continuous functions $\lambda(t)$ from $\mathbb{R}$ to itself with $\lambda(0) = 0$.

The Borel $\sigma$-algebra under this topology is denoted as $\mathcal{F}$. Moreover, for every $t \in \mathbb{R}$, a stochastic process is also a random variable in the following sense,

$$L_t(\omega) : \omega \to D(\mathbb{R}, E) \quad \omega \to L(t), \ t \in \mathbb{R}.$$

**Definition.** *The probability measure, $\mathbb{P}$ in $(D(\mathbb{R}, E), \mathcal{F})$ that makes every element in $D(\mathbb{R}, E)$ a sample Lévy path is called the Lévy probability measure. Note, this measure shall not be confused with the Lévy measure for jumps.*

Now define the flow $\vartheta = (\vartheta_t, t \in \mathbb{R})$ on this canonical sample space $\Omega$ by the shift

$$(\vartheta_t \omega)(\cdot) = \omega(t + \cdot) - \omega(t) \quad t \in \mathbb{R}, \omega \in \Omega.$$

The Lévy probability measure $\mathbb{P}$ is invariant under this flow, that is,

$$\mathbb{P}(\vartheta_t^{-1}(T)) = \mathbb{P}(T)$$

for all $T \in \mathcal{F}$. This flow is an ergodic dynamical system with respect to $\mathbb{P}$. Thus $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta)_{t \in \mathbb{R}})$ is a metric DS.

*Step 2* Prove that the Ornstein Uhlenbeck process in the metric dynamical system is also well defined.

**Proposition 4.3.1.** *Assume that $L$ is a Lévy process taking value in $E$, such that for any $T > 0$*

$$\int_0^T |L(t)|_E dt < \infty.$$

*Then the solution of the differential equation*

$$\dot{V}(t) + \alpha V(t) = L(t), \quad V(0) = 0,$$

*has the form*

$$V(t) = \int_0^t e^{-(t-s)A} L(s) ds.$$

Using the above expression, combine with the integration by part technique from Xu[113], one can represent the Ornstein Uhlenbeck process in the metric dynamical system as

$$z(\vartheta_t \omega) := \hat{z}(t) = \int_{-\infty}^t A e^{-(t-r)A}(\tilde{\omega}(t) - \tilde{\omega}(r)) dr, \quad t \in \mathbb{R}.$$

We show $\hat{z}(t)$ is a well defined element in $X := \mathbb{L}^4(\mathbb{S}^2) \cap H$ by establishing Proposition 4.3.2

**Proposition 4.3.2.** *Assume $0 < \delta < \frac{1}{\beta}$, $\beta \in (1, 2)$, $p \in (0, \beta)$ and*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty.$$

*Then*

$$\mathbb{E} \int_{-\infty}^t |\hat{A} e^{-(t-r)\hat{A}}(\tilde{\omega}(t) - \tilde{\omega}(r))|_X^p dr < \infty.$$

*Moreover, for $\mathbb{P}$* almost every $\tilde{\omega} \in D(\mathbb{R}, X)$, $t \in \mathbb{R}$ *the function*

$$\hat{z}(t) = \hat{z}(\tilde{\omega})(t) = \int_{-\infty}^t \hat{A}^{1+\delta} e^{-(t-r)\hat{A}}(\tilde{\omega}(t) - \tilde{\omega}(r)) dr, \quad t \in \mathbb{R}.$$

*is well defined and cádlág in $X$. Furthermore, there exists a constant $C$ depending on $\beta$, $p$, $\sigma$, $\delta$, $\omega$ such that*

$$|\hat{z}(\tilde{\omega})(t)| \leq C(\beta, p, \sigma, \delta, \tilde{\omega})(1 + |t|^\delta).$$

The proof is quite involved, we refer readers to 4.3.2.2.

**Theorem 4.3.3.** *Under the assumption of Proposition 4.3.2, for $\mathbb{P}$-a.e. $\omega \in D(\mathbb{R}, X)$,*

$$\hat{z}(\vartheta_s\omega)(t) = \hat{z}(\omega)(t+s), \quad t, s \in \mathbb{R}.$$

**Proposition 4.3.4.** *We define*

$$z_\alpha(t) = \int_{-\infty}^{t} e^{-(t-s)(\hat{A}+\alpha I)} dL(s)$$

*where the integral is intepreted in the sense of [23] is well defined and is identified as a solution to the equation*

$$dz_\alpha(t) + (\hat{A} + \alpha I) z_\alpha dt = dL(t), \quad t \in \mathbb{R}.$$

*The process $z_\alpha$, $t \in \mathbb{R}$ is a stationary OU process.*

Under Proposion 4.3.4, we define

$$z_\alpha(\omega) := \hat{z}(\hat{A} + \alpha I; (\hat{A} + \alpha I)^{-\delta}\omega) \in D(\mathbb{R}, X).$$

It follows from Proposition 4.3.1 that for any $t > 0$

$$z_\alpha(\omega)(t) := \int_{-\infty}^{t} (\hat{A} + \alpha I)^{1+\delta} e^{-(t-r)(\hat{A}+\alpha I)} [(\hat{A} + \alpha I)^{-\delta}\omega(t) - (\hat{A} + \alpha I)^{-\delta}\omega(s)] ds = L(t). \tag{1.30}$$

Moreover, by the fundamental theorem of calculus, one has,

$$\frac{dz_\alpha}{dt} + (\hat{A} + \alpha I) \int_{-\infty}^{t} (\hat{A} + \alpha I)^{1+\delta} e^{-(1-r)(\hat{A}+\alpha I)} [(\hat{A} + \alpha I)^{-\delta}\omega(t) - (\hat{A} + \alpha I)^{-\delta}\omega(r)] dr = \dot{\omega}(t).$$

Therefore $z_\alpha(t)$ satisfies

$$\frac{d^+}{dt} z_\alpha(t) = (\hat{A} + \alpha I) z_\alpha = \eta(t), t \in \mathbb{R} \tag{1.31}$$

where $\eta(t) := \eta(t, x)$ is the Lévy white noise defined in Chapter 3. Hence Theorem 4.3.3 yields

$$\hat{z}(\vartheta_s\omega)(t) = \hat{z}(\omega)(t+s), \quad t, s \in \mathbb{R} \quad \omega \in D(\mathbb{R}, X), \ t, s \in \mathbb{R}.$$

Similar to our definition of Lévy process

$$L_t(\omega) := \omega(t), \quad \omega \in \Omega.$$

The formula (1.30) defines the process $z_\alpha(t)$, $t \in \mathbb{R}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equation (1.31) suggests $z_\alpha$ is a Qrnstein Uhlenbeck process.

*Step 3* Prove that $(\varphi, \vartheta)$ defines a RDS.

The proof of step 3 is accomplished by checking *measurability, continuous dependence on initial data, and the Cocycle property.* The argument follows the same lines as the Wiener case [18].

**Theorem 4.3.5.** $(\varphi, \vartheta)$ *is a random dynamical system.*

#### 1.3.4.3 Existence of Random Attractors

The last subsection 4.3.4 is devoted to the existence of a random attractor which supports a markov invariant measures.
Having found a suitable sample probability space which ensures the linear stochastic Stokes equation remains stationary, in subsection 4.3.4 we prove the existence of random attractors. Our methodology was inspired by Flandoli and Crauel[34]. In brief, we use the following a priori estimates for a strong solution (bounded in $V$, compact in $H$).

**Lemma 4.3.6.** *Suppose that $v$ is a solution to problem (3.75) on the time interval $[a, \infty)$ with $z \in L^4_{loc}(\mathbb{R}, \mathbb{L}^4(\mathbb{S}^2)) \cap L^2_{loc}(\mathbb{R}, V')$ and $\alpha \geq 0$. Then, for any $t \geq \tau \geq a$, one has*

$$|v(t)|^2 \leq |v(\tau)|^2 e^{\int_\tau^t \gamma(s)ds} + \int_{t_0}^t e^{\int_s^t \gamma(\xi)d\xi} 2p(s)ds, \tag{1.32}$$

*where*

$$p(t) = c|f|^2 + c\alpha|z|^2 + \eta|z|^2 \sum_{k=1}^m |z_k(t)|, \tag{1.33}$$

$$\gamma(t) = -\frac{\lambda_1}{2} + 4\beta \sum_{k=1}^m |z_k(t)| \tag{1.34}$$

*for all $t_0 \leq \tau \leq t$ and $c$ depends only on $\lambda_1$.*

Using this result, we prove, respectively the existence of an absorbing balls in $H$ at $t = -1$ (Lemma 4.3.9) and in $V$ at $t = 0$ (Lemma 4.3.10). Then with the use of the compact embedding of Sobolev spaces, we identified a compact absorbing set and consequently deduce the existence of a random attractor.
Finally, it follows from the following corollary 4.4 Flandoli and Crauel [34],

**Corollary.** *If a càdlàg time and space continuous RDS $\varphi$ has an invariant compact random set, then there exists a feller invariant probability measure $\mu$ for $\varphi$.*

that the random attractor supports a random invariant measure.
In relation to the literature of random attractors, on the one hand our study generalises to the

case of Lévy noise, several earlier results on random attractors in the case of Gaussian noise [15, 21, 34]. It also generalises to the case of 2D sphere with rotation, the results [21, 34] for the case of bounded or unbounded domain in $\mathbb{R}^2$.

#### 1.3.4.4 Contributions

In view of the above, the main contributions of Chapter 5 are:

- Construction of a new canonical sample space for the SNSE with Lévy noise;
- A new regularity result on the dynamics of the driving noise (Proposition 5.3.2);
- Existence of a RDS $\varphi$ (Theorem 5.3.5);
- Existence of a random attractor (See section 5.3.4);
- Existence of a Feller Markov Invariant Measure (See section 5.3.5).

# CHAPTER 2

## Subordinated $H$-cylindrical Wiener processes and Stochastic Integration

## Summary

In this chapter, we start with a review of probability and analytic facts. Then we introduce the concept of infinite divisibility of probability distribution and random variables which underpins the whole subject of Lévy Processes. The Lévy Itô decomposition describes the structure of their sample paths while the Lévy Khintchine formula prepares one to study distributional properties of Lévy process. Lévy processes are introduced in Section 2.3. In particular we revisit the concepts of real-valued Wiener process, then the Hilbert space-valued Wiener process and and the cylindrical Wiener process. The notion of Reproducing Kernel Hilbert Space is introduced as well. Finally, the Hilbert space valued (or $H$- valued) Lévy process and $L^p$-valued Lévy process are defined. In section 2.4 we review stochastic integration in Hilbert space. In particular, we present a version of the stochastic Fubini theorem for general Lévy process which seems to be new. Section 2.5 introduces concepts of subordinators and subordinated processes that are naturally defined via convolution semigroups of probability measures. In the final subsubsection, we define cylindrical stable process as a subordinated cylindrical Wiener process.

## 2.1 Preliminaries

In this section we review some basic probabilistic and analytic tools to aid the core discussion in Chapter 3 and 4.

### 2.1.1 Probabilistic preliminaries

In this section we recall some basic facts from probability theory in the context of random dynamical systems. The discussion follows closely to the material in [32, 37, 95].

Let $E$ be a non-empty set and $\mathcal{E}$ a collection of subsets of $E$. We call $\mathcal{E}$ a $\sigma$-algebra if the following hold

- $E \in \mathcal{E}$.
- $A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$
- $A_n \in \mathcal{E} \Rightarrow \cup_{n=1}^{n} A_n \in \mathcal{E}$.

The pair $(E, \mathcal{E})$ is called a *measurable space*. A measure on $(E, \mathcal{E})$ is a mapping $\mu : \mathcal{E} \to [0, \infty]$ that satisfies

- $\mu(\emptyset) = 0$,
- $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$
  for every sequence $(A_n, n \in \mathbb{N})$ of mutually disjoint sets in $\mathcal{E}$.

The triple $(E, \mathcal{E}, \mu)$ is called a *measure space*. We will frequently use the notation $E = \Omega$ and will interpret $\Omega$ as the set of outcomes of some random experiments. Elements of $\mathcal{E}$ are called *events* and any measure on $(\Omega, \mathcal{E})$ of total mass 1 is called a *probability measure* will be denoted as $\mathbb{P}$.
Le $E$ be a Polish space. The Borel $\sigma$-algebra of $E$ is the smallest $\sigma$-algebra containing all closed (or open) subsets of $E$ and it is denoted as $\mathcal{B}(E)$. A function $Y : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is said to be $\mathcal{F}$-measurable if for any set $B \in \mathcal{E}$

$$Y^{-1}(B) = \{\omega : Y(\omega) \in B\} = \{Y \in B\} \in \mathcal{F}$$

So, any $E$-valued r.v. $Y$ is a measurable function $Y : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$. Now, Let $I \subset \mathbb{R}^+$ be some set of indices, then an $E$-valued stochastic process is a family $\{Y(t)\}_{t \in I}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A collection of probabilities $\{\mathbb{P}(Y \in B); B \in \mathcal{B}(E)\}$ is called the *probability law*, denoted as $\mathcal{L}(Y)$.
Let $I$ be a time interval. In general, $I$ can be taken as the set of non-negative real number $\mathbb{R}^+$, or a finite interval $[0, T]$ or, in the discrete-time case, a subset of non-negative integers $\mathbb{Z}_+ = \{0, 1, \cdots\}$. On $I$ we consider the Borel $\sigma$-field $\mathcal{B}(I)$. Any family $Y = (Y(t), t \in I)$ of random variables in $E$ is called a *stochastic process* in $E$ or *$E$-valued stochastic process*.
There are several form of continuity one shall keep in mind. We say that $X$ is *continuous* (respectively càdlàg) if for $\mathbb{P}$-a.e. $\omega \in \Omega$, the sample-paths $t \mapsto Y(t, \omega) \in E$ of $Y$ are continuous. $X$ is said to be càdlàg if $X$ is right-continuous with left limits (RCLL). That is, for every $t \in I$, the limit $\lim_{s \in I, s \uparrow t} X(s) = X(t-)$ exists.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *filtration* is any non-decreasing family of $\sigma$-fields, that is, $\mathcal{F}_s \subseteq \mathcal{F}_t$ for any $0 \le s \le t \le T$. $\mathcal{F}_t \subset \mathcal{F}$, $t \in I$. The filtration $(\mathcal{F}_t)_{t \ge 0}$ is abbreviated as $\mathbb{F}$. A probability space is said to be filtered if it is equipped with a filtration $\mathbb{F}$. A filtration $(\mathcal{F}_t)$ is

said to be right-continuous if

$$\mathcal{F}_t = \mathcal{F}_{t+} = \bigwedge_{s>t} \mathcal{F}_s$$

We will use standard notation

$$\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$$

This is the $\sigma$-field of events strictly prior to time $t$. The quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_{t\geq 0}), \mathbb{P})$ is called a *filtered probability space*. We say that an $E$-valued stochastic process $Y$ is *adapted to the filtration* $(\mathcal{F}_t)$ if, for every $t \in I$, $Y(t)$ is $\mathcal{F}_t$-measurable.

The *Dirac measure* on $E$ for a given $x \in E$ will be denoted by $\delta_x$ and is defined by $\delta_x(A) = 1_A(x)$, for any $A \subseteq E$.

Let $\text{Pr}(E)$ be the space of all Borel probability measures on $E$ endowed with the topology of weak convergence of measures.

The notion of *tightness* plays an important role in this thesis. Denote $\text{Pr}(E)$ as the set of probability measures on $E$. Recall, a set $G \subset \text{Pr}(E)$ is said to be *tight* if for every $\epsilon > 0$ there exists compact set $K_\epsilon \subset E$ such that $\mu(K_\epsilon) > 1 - \epsilon$ for all $\mu \in G$ (This is also called "uniformly tight").

The Prohorov's Theorem gives a criterion for a sequence probability distribution to possess a weakly convergent subsequence.

**Theorem 2.1.1.** *Let $E$ be a Polish space and Let $G$ be a subset of $Pr(E)$. Then $\bar{G}$ is compact*[1] *in $Pr(X)$ if and only if $G$ is tight.*

**Proof.** See Parthasarathy [85]. ☐

**Definition 2.1.2** (Modification)**.** Let $X := (X_t)_{t\geq 0}$ and $Y := (Y_t)_{t\geq 0}$ be two $E$-valued stochastic processes defined on the same probability space. We say that $X$ is a *modification* of $Y$ if $\mathbb{P}(\{X_t = Y_t\}) = 1$ for all $t \geq 0$.

In what follows we will need a convolution of two probability measures defined as

$$(\mu_1 * \mu_2)(A) = \int_E \mu_1(A - x)\mu_2(dx)$$

for each $A \in \mathcal{B}(E)$, where $A - x = \{y - x, y \in A\}$. Denote by $\nu^{*n} = \nu * \cdots \star \nu$ ($n$-times) with $\nu^{\star 0} = \delta_0$.

[1]That is, for any $\{u_n\} \subset G$, there is a weakly convergent subsequence.

#### 2.1.1.1 Random Probability Measures

Let us quote some standard facts from measure theory. The discussion in this subsection follows closely the book [32].

Let $X$ be a Polish space with a metric $d$. The $\sigma$-field of Borel sets of $X$ is denoted by $\mathcal{B}$. The product space $X \times \Omega$ is understood as a measurable space with the product $\sigma$-field $\mathcal{B} \otimes \mathcal{F}$, which is the smallest $\sigma$-field on $\Omega \times X$ with respect to which both the canonical projection $\pi_X : \Omega \times X \to X$ and $\pi_\Omega : \Omega \times X \to \Omega$ are measurable.

**Definition 2.1.3** (Random Probability Measure)**.** A map

$$\mu : \mathcal{B} \times \Omega \to [0, 1],$$
$$(B, \omega) \mapsto \mu_\omega(B)$$

satisfying

(i) for every $B \in \mathcal{B}, \omega \mapsto \mu_\omega(B)$ is measurable,

(ii) for $\mathbb{P}$-almost every $\omega \in \Omega$, $B \mapsto \mu_\omega(B)$ is a Borel probability measure on $X$.

is said to be a *random probability measure* on $X$, and is denoted by $\omega \mapsto \mu_\omega$.

Random probability meausre are also called *transition probabilities* or *Markov kernels*. Let $\mu_\cdot(\cdot)$ be a transition probability from $\Omega$ to $X$, i.e., $\mu_\omega$ is a Borel probability measure on $X$ and $\omega \mapsto \mu_\omega(B)$ is measurable for every Borel set $B \subset X$. Denote by $\mathrm{Pr}_\Omega(X)$ the set of transition probabilities with $\mu\cdot$ and $\nu\cdot$ identified if $\mathbb{P}(\omega : \mu_\omega \neq: \nu_\omega) = 0$.

Let us now consider the space of all probability measures on $(\Omega \times X, \mathcal{B} \times \mathcal{F})$ with marginal $\pi_\Omega\mu = \mathbb{P}$ on $\Omega$. The following proposition about disintegration relates random measure $\omega \mapsto \mu_\omega$ with its associated marginal $\mathbb{P}$ measure on the product space $\Omega \times X$.

**Proposition 2.1.4** (Existence and Uniqueness of a Disintegration, p.19[32])**.** *For every probability measure $\mu$ on $\Omega \times X$ with $\pi_\Omega\mu = \mathbb{P}$ there exists a random measure $\omega \mapsto \mu_\omega$ such that*

$$\int_{\Omega\times X} h(\omega, x) d\mu(\omega, x) = \int_X \int_\Omega h(\omega, x) d\mathbb{P}(\omega) d\mu_\omega(x)$$

*for every bounded measurable $h : \Omega \times X \to \mathbb{R}$. The random measure $\omega \mapsto \mu_\omega$ is unique $\mathbb{P}$-a.s..*

The two random measures $\omega \mapsto \mu_\omega$ and $\omega \mapsto \nu_\omega$ coincide if $\mu_\omega = \nu_\omega$ for $\mathbb{P}$ almost all $\omega$. Put

$$\mathrm{Pr}_\Omega(X) = \{\mu : \mathcal{B} \times \Omega \to [0, 1] : \omega \mapsto \mu_\omega \text{ random measure}\}$$

with two random measures identified if they coincide $\mathbb{P}$-a.s., and

$$\mathrm{Pr}_\mathbb{P}(\Omega \times X) = \{\mu \in \mathrm{Pr}(\Omega \times X) : \pi_\Omega\mu = \mathbb{P}\}.$$

In view of the above Proposition 2.1.4, suppose $\mu$ is a probability measure on $\Omega \times X$ with marginal $\mathbb{P}$ on $\Omega$. Then for any $\mu \in \Pr_{\mathbb{P}}(\Omega \times X)$ there is a disintegration $\mu_{\cdot} \in \Pr_{\Omega}(X)$ uniquely determined by

$$\mu(B \times F) = \int_F \mu_\omega(B) d\mathbb{P}(\omega)$$

for all $B \in \mathcal{B}$ and $F \in \mathcal{F}$. With this one can identify probability measures on $\Omega \times X$ with marginal $\mathbb{P}$ with their disintegration $\omega \mapsto \mu_\omega$.

**Definition 2.1.5.** A probability measure $\mu$ on $\Omega \times X$ with marginal $\mathbb{P}$ on $\Omega$ is said to be *supported by a measurable random set* $\omega \mapsto A(\omega)$, if $\mu(A) = 1$, where $A = \{(\omega, x) : x \in A(\omega)\}$ is the graph of the mapping $\omega \mapsto A(\omega)$. Equivalently, $\mu_\omega(A(\omega)) = 1$ $\mathbb{P}$-a.s..
Denote by $C_\Omega(X)$ the set of function $f : X \times \Omega \to \mathbb{R}$ such that $f(\cdot, x)$ is measurable for each $x \in X$, $f(\omega, \cdot)$ is continuous and bounded for each $\omega \in \Omega$, and $\omega \mapsto \sup\{|f(\omega, x)| : x \in X\}$ is integrable w.r.t. $\mathbb{P}$, where two such functions $f$ and $g$ are identified if $\mathbb{P}(\{\omega : f(\omega, \cdot) \neq g(\omega, \cdot)\}) = 0$ (measurable by continuity of $f$ and $g$ together with separability of $X$). Define the *narrow topology* on $\Pr_\Omega(X)$ to be the coarest topology such that

$$\mu \mapsto \int_{\Omega \times X} f(\omega, x) d\mu(\omega, x) = \mu(f)$$

is continuous for all $f \in C_\Omega(X)$. The skew product flow $\{\Theta_t\}_{t \in T}$ acts as a flow of continuous transformations on $\Pr_\Omega(X)$.
A generalisation of the Prohorov theorem for random measures is the following.

**Definition 2.1.6** (Tightness for random measures)**.** A subset $\Gamma$ of $\Pr_\Omega(X)$ is said to be *tight*, if for every $\epsilon > 0$ there exists a compact set $C_\varepsilon \subset X$ such that $(\pi_X \gamma)(C_\epsilon) \geq 1 - \epsilon$ for every $\gamma \in \Gamma$, where $\pi_X \gamma : X \times \Omega \to X$ is the canonical projection (see p.31 [32]).

**Theorem 2.1.7** ([108])**.** *Suppose $\Gamma \subset Pr_\Omega(X)$ is tight. Then*

- *$\Gamma$ is relatively compact in $Pr_\Omega(X)$.*
- *$\Gamma$ is relatively sequentially compact (i.e. if $\{\mu_{\cdot}^n\}_{n \in \mathbb{N}}$ is a sequence in $\Gamma$, then there exists a convergent subsequence $\{\mu_{\cdot}^{n_k}\}_{k \in \mathbb{N}}$)*

#### 2.1.1.2 Skorohod space and Skorohod metric

In this subsection we recall some basic terminology on Skorohod space and Skorohod metric given in section 12 of the book [12].

Let $E$ be a separable Banach space with the norm $|\cdot|$. Let us denote by $D(\mathbb{R}, E)$ the space of all functions defined on $\mathbb{R}$ with values in $E$ that are right continuous and have left limits (càdlàg paths). Let $\Lambda$ denote the class of strictly increasing, continuous mappings $\lambda(t)$ of $\mathbb{R}$ into itself with $\lambda(0) = 0$. One introduces a topology on $D(\mathbb{R}, E)$ by defining the Skorokhod metric as

$$d(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i}(1 \wedge d_i^\circ(f, g))$$

where

$$d_i^\circ(f, g) := \inf_{\lambda \in \Lambda} \left\{ \sup_{-i \le s < t \le i} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \vee \sup_{-i \le < t \le i} |f(t) - g(\lambda(t))| \right\}$$

The space $D((-\infty, \infty), E)$ endowed with the metric $d$ is complete and separable.

### 2.1.2 Analytic preliminary

In this subsection we recall some basic facts in the theory of analytic semigroup.

#### 2.1.2.1 Semigroups and analytic Semigroups

Let $X$ be a Banach space. The space of bounded linear operators is denoted as $\mathcal{L}(X)$. A family of operators $S(t) \in \mathcal{L}(X)$ with $t > 0$ is a one-parameter *semigroup* if $S(t_1 + t_2) = S(t_1)S(t_2)$ for $t_1, t_2 > 0$ and $S(0) = I$. Moreover, if

$$t \mapsto S(t)x$$

is continuous for every $x$, then $(S(t))_{t \ge 0}$ is said to be *strongly continuous* or shortly a $C_0$-semigroup. In particular, every $A \in \mathcal{L}(X)$ generates a strongly continuous semigroup $S(t) = \mathrm{e}^{-tA}$ where $\mathrm{e}^{-tA} := I - tA + \frac{1}{2}t^2A^2 - \cdots$. In general, the *infinitesimal generator* of a semigroup $(S(t))_{t \ge 0}$ is defined as the operator

$$Ax := \lim_{t \downarrow 0} \frac{S(t)x - x}{t}$$

whose domain $D(A)$ is defined as

$$D(A) := \left\{ x \in X : \lim_{t \downarrow 0} \frac{(S(t) - I)x}{t} \text{ exists} \right\}$$

The resovent set of $A$ is the set $\rho(A)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists a unique bounded linear operator $R(\lambda, A)$ on $X$ such that

- $R(\lambda, A)(\lambda I - A)x = x$ for all $x \in D(A)$;
- $R(\lambda, A)x \in D(A)$ and $(\lambda I - A)R(\lambda, A)x = x$ for all $x \in X$

Explicitly, the resolvent set of $A$ is defined as

$$\rho(A) := \{\lambda \in \mathbb{C}; \lambda I - A : D(A) \to X \quad \text{bijective}, \quad (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$$

The operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is called the resolvent of $A$ at $\lambda$, and the mapping

$$R(\lambda, A) : \rho(A) \to \mathcal{L}(X)$$

is called the resolvent of $A$ at $\lambda$, and the mapping

$$R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$$

is called the resolvent of $A$. The set

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is said to be the spectrum of $A$.
For $\sigma \in (0, \pi]$ we define the open sector

$$\Sigma_\sigma = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \sigma\}$$

where the argument is taken in $(-\pi, \pi]$. A $C_0$ semigroup $(S(t), t \geq 0)$ is called analytic on $\Sigma_\sigma$ if

- $S$ extends to an analytic function $S : \Sigma_\sigma \to \mathcal{L}(X)$;
- $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \Sigma_\sigma$;
- $\lim_{z\to 0; z\in\Sigma_\sigma} S(z)x = x$ for all $x \in X$.

An equivalent definition in [87] is the following

**Definition 2.1.8.** A $C_0$-semigroup on a Banach space $X$ is *analytic* if, for every $t > 0$, $S(t)(X) \subset D(A)$ and $\sup_{t\in(0,1]} |tAS(t)|_{L(X,X)} < \infty$

#### 2.1.2.2 Fractional powers of operators

To study the regularity properties of solutions to equations with a linear part $A$, that is, the generator of an analytic semigroup $S$ on a Banach space $X$, it is convenient to introduce the concept of *fractional power* of $A$. Roughly speaking, for any infinitesimal generator where its spectrum does not contain zero and does not surround it, $A^\alpha$ is defined by a Cauchy integral along a contour around the spectrum does not containing 0.
Assume that there exists $\epsilon > 0$ and $0 < M < \infty$ such that

$$|(\lambda I - A)^{-1}|_{\mathcal{L}(X)} \leq \frac{M}{\epsilon + \lambda}, \quad \forall \lambda > 0 \tag{2.1}$$

Then we can define a semigroup $S(\cdot)$ of bounded linear operator in $X$ by setting $S(0) = I$ and

$$S(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{\lambda} R(\lambda, A)d\lambda, \quad t > 0$$

For any $\gamma > 0$, the bounded linear operator $A^{-\gamma}$ is defined by

$$A^{-\gamma} := \frac{1}{2\pi i}\int_{\mathcal{C}} \lambda^{-\gamma} R(\lambda, A)d\lambda$$

where $\mathcal{C}$ is a piecewise smooth path in $\rho(A)$ from $\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$ for some $\vartheta > 0$. The integral above converges in the uniform operator topology for every $\gamma > 0$ and thus defines a bounded linear operator $A^{-\gamma}$. For $0 < \gamma < 1$ one can deform the path of integration $\mathcal{C}$ into the upper and lower sides of the negative real axis and obtain

$$A^{-\gamma} = \frac{1 - e^{-2\pi i\gamma}}{2\pi i}\int_0^{\infty} \lambda^{-\gamma}(\lambda I - A)^{-1} d\lambda$$

One can check that $A^{-\gamma}$ is injective for every $\gamma > 0$ (See Nigel [46], Prop 5.30). We are now ready to define the fractional power of $A$.

**Definition 2.1.9.** Let $\gamma > 0$. Then the operator $A^{\gamma}$ defined as the inverse of $A^{-\gamma}$ with domain $D(A^{\gamma}) = \text{ran}(A^{-\gamma})$ is called the $\gamma$-power of $A$.

Let $U$, $H$ and $K$ be three real separable Hilbert spaces. We denote by $L(U, H)$ the space of all bounded linear operators from $U$ to $H$.

**Definition 2.1.10.** A Linear operator $R \in L(U, H)$ is called *Hilbert-Schmidt* if

$$\sum_k |Re_k|_H^2 < \infty$$

for any (or equivalently for a certain) orthonormal basis $\{e_k\}$ in $U$.

It is often convenient to introduce the concept of so-called $\gamma$-radonifying operators. To do this, fix an orthonormal basis $\{e_j\}$ of some Hilbert space $K$. Let $\{\sigma_j\}$ be a sequence of independent standard Gaussian distributed real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.1.11.** A bounded linear operator $\Psi : K \to X$ is called *$\gamma$-radonifying* if the series $\sum_{j=1}^{\infty} \sigma_j \Psi e_j$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; X)$

Finally, in the following we recall some well known probabilistic and analytic facts from the seminal paper [14]. Let $\mathcal{H}$ and $\mathcal{E}$ be respectively real separable Hilbert and Banach spaces. Let $\{e_k\}$ be a fixed orthonormal basis of $H$. Let $\mathcal{B}_0(K)$ denotes the class of all subsets $\mathcal{U}$ of $K$

having the form

$$\mathcal{U} = \{v \in K : (\langle v, h_1\rangle, \cdots, \langle v, h_n\rangle) \in \mathcal{U}_0\}$$

for a certain $n$, an orthonormal system $h_1, \cdots, h_n$ in $K$, and $\mathcal{U}_0 \in \mathcal{B}(\mathbb{R}^n)$. Let $\gamma_K$ be a standard cylindrical Gaussian measure on a real separable Hilbert space $K$. [2] A bounded linear operator $U : K \hookrightarrow X$ is $\gamma$-radonifying iff the image $U(\gamma_K)$ of the canonical Gaussian distribution $\gamma_K$ on $K$ extends to a Borel (Gaussian) Probability measure on $X$ which will be denoted by $\nu_U = \gamma_K \circ U^{-1}$. Set

$$R(K, X) := \{U : K \to X \quad \text{such that} \quad U \text{ is } \gamma \text{ radonifying}\}$$

By the Fernique theorem (which asserts that for any $U \in R(K, X)$ there exists a $c > 0$ such that $\int_X e^{c|x|^2} \nu_U(dx) < \infty$ ) , one can equip the space $R(K, X)$ with the norm

$$|U|_{R(K,X)} := \left( \int_X |x|_X^2 \nu_K(dx) \right)^{1/2}.$$

Let $\mathbb{S}^2$ be the 2D unit sphere. We denote $\int_{\mathbb{S}^2} f dS$ the integration with respect to the surface measure $dS = \sin\theta d\theta d\phi$.
Let $B : H \to H$ be a selfadjoint operator with the complete orthonormal system of eigenfunctions $(e_l) \subset L^p(\mathbb{S}^2)$ and the corresponding set of eigenvalues $(\lambda_l)$. It follows from from Theorem 2.3 [22] that if further $B$ has compact inverse $B^{-1}$ then the operator $U^{-s} : H \to L^p(\mathbb{S}^2)$ is well defined and $\gamma$-radonifying iff

$$\int_{\mathbb{S}^2} \left( \sum_l \lambda_l^{-2s} |e_l(x)|^2 \right)^{p/2} dS(x) < \infty \tag{2.2}$$

## 2.2 Basic properties: Infinite divisibility

Let $\mu^{*n}$ denotes the $n$-fold convolution of probability measure $\mu$ with itself. A probability measure $\mu$ on a separable Hilbert space $H$ is called infinitely divisible if for every $n \geq 1$ there exists a probability measure $\mu_n$ such that $\mu = \mu_n^{*n}$.
Every infinitely divisible distribution can be uniquely determined in term of characteristic function. Let $\mu$ be an infinitely divisible distribution on $H$. Then, its characteristic function $\hat{\mu}$ is given for every $x \in H$ by

$$\hat{\mu}(x) = \exp\left\{ i(x, b) - \frac{1}{2}(x, Qx) + \int_H (e^{i(x,y)} - 1 - i(x, y)\chi_{|y|\leq 1}(y))\nu(dy) \right\}, \tag{2.3}$$

[2]That is, $\gamma(U) = (2\pi)^{-n/2} \int_{U_0} \exp(-|x|^2/2) dx$ for $U$.

where $b \in H$, $Q$ is a nonnegative trace class operator on $H$ and $\nu$ is a measure concentrated on $H \setminus \{0\}$ satisfying the condition

$$\int_H (|y|^2 \wedge 1)\nu(dy) < \infty.$$

Moreover, the Lévy triplet $(b, Q, \nu)$ is uniquely determined by the measure $\mu$. Conversely, every Lévy triplet $(b, Q, \nu)$ uniquely determines an infinitely divisible distribution with characteristic function (2.3). This result leads to the *Lévy Khintchine representation* or *Lévy Khintchine formula* which we will state in section 2.2.1.

Let $\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that a random variable $X : \Omega \to H$ is *infinitely divisible* if, for every $n \in \mathbb{N}$, there exist *i.i.d.* $H$-valued random variables $Y_1^{(n)}, \cdots, Y_n^{(n)}$ such that

$$X \overset{\mathrm{d}}{=} Y_1^{(n)} + \cdots + Y_n^{(n)}.$$

### 2.2.1 Lévy Khintchine formula

We have the following Lévy Khintchine formula [87]

**Theorem 2.2.1** (Lévy Khintchine)**.** *Given $b \in H$, $Q \in L_1^+(H)$ and a non-negative measure $\nu$ concentrated on $H \backslash \{0\}$ satisfying*

$$\int_H (|y|_H^2 \wedge 1)\nu(dy) < \infty, \tag{2.4}$$

*there exists a convolution semigroup $(\mu_t)$ of measures on $H$ such that*

$$e^{-t\psi(x)} = \int_H e^{i\langle x,y\rangle_H} \mu_t(dy), \tag{2.5}$$

*where*

$$\psi(x) = -i\langle b, x\rangle_H + \frac{1}{2}\langle Qx, x\rangle_H + \int_H (1 - e^{i\langle x,y\rangle_H} + \chi_{\{|y|_H<1\}}(y) i\langle x, y\rangle_H)\nu(dy). \tag{2.6}$$

*(b) Conversely, for each convolution semigroup $(\mu_t)$ of measures, there exists $b \in H$, $Q \in L_1^+(H)$ and a non-negative measure $\nu$ concentrated on $H \backslash \{0\}$ satisfy (2.4) in such a way that (2.5) holds with $\psi$ defined by (2.6).*

## 2.3 Lévy Process

In this Section, our aim is to present some basic facts in the theory of $H$-valued Lévy process. The presented material follows closely to the books [38, 87]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a given filtered probability space and all stochastic processes are defined on this space.

### 2.3.1 Wiener process

A real-valued $(\mathcal{F}_t)$-adapted stochastic process $(W(t), t \geq 0)$ is said to be a Wiener process if

(i) $W$ has continuous trajectories with $W(0) = 0$,

(ii) $W$ has independent increments and

$$\mathcal{L}(W(t) - W(s)) = N(0, t - s), \quad t \geq s \geq 0,$$

where $N(m, \sigma^2)$ stands for the Gaussian distribution with mean $m$ and variance $\sigma^2$. Equivalently, a real valued stochastic process $(W(t))$ with continuous trajectories is called a Wiener process if it is Gaussian and there exists $\sigma \geq 0$ such that

$$\mathbb{E}(W(t)) = 0, \quad \mathbb{E}(W(t)W(s)) = \sigma^2(t \wedge s).$$

### 2.3.2 Hilbert space valued Wiener process and cylindrical Wiener process

Let $H$ be a real separable Hilbert space with inner product $(\cdot, \cdot)$. An $H$-valued, $(\mathcal{F}_t)$-adapted Wiener process is such a process that for every $x \in H$ with $|x| = 1$, a real-valued process $(W(t), x)$, $t \geq 0$, is an $(\mathcal{F}_t)$-adapted Wiener process. This implies in particular that the law $\mathcal{L}(W(t))$ of $W(t)$, is a Gaussian measure with mean vector 0 and, for arbitrary $x, y \in H$, $t, s \geq 0$,

$$\mathbb{E}[(W(t), x)(W(s), y)] = (t \wedge s)\mathbb{E}[(W(1), x)(W(1), y)] = (t \wedge s)(Qx, y),$$

where $Q$ is the covariance operator of the Gaussian measure $\mathcal{L}(W(1))$. The operator $Q$ is a symmetric trace class and positive, that is, for any orthonormal basis $\{e_j : j \geq 1\}$ of $H$ the operator $Q$ has the property

$$\operatorname{Tr} Q = \sum_{j=1}^{\infty}(Qe_j, e_j) < \infty,$$

and $(Qx, x) \geq 0$ for any $x \in H$. The Wiener process defined this way is sometimes called a $Q$-Wiener process to emphasise its dependence on the covariance operator $Q$.
Finally, we will define a cylindrical Wiener that intuitively corresponds to the case $Q = I$.

More precisely, by an $(\mathcal{F}_t)$-adapted cylindrical Wiener process on $H$ we mean a mapping $W : [0,\infty] \times H \mapsto L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following conditions.

- for every $t \geq 0$ and $x \in H$ the random variable $W(t,x)$ is $\mathcal{F}_t$-measurable,
- for every $t \geq 0$ the mapping $H \ni x \mapsto W(t,x) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is linear,
- for all $t \geq 0$, and $x, y \in H$, $\mathbb{E}|W(t,x)|^2 = t|x|_H^2$,
- for each $x \in H$ with $|x| = 1$, $W(t,x)$, $t \geq 0$ is a real valued Wiener process.

**Lemma 2.3.1.** *If $W$ is a cylindrical Wiener process then, for all $t \geq s \geq 0$ and $x, y \in H$,*

$$\mathbb{E}W(t,x)W(s,y) = (t \wedge s)\langle x, y\rangle_H.$$

### 2.3.3 Reproducing Kernel Hilbert Space

Let us now present some additional facts about reproducing Hilbert spaces and Banach space-valued Wiener processes. We will follow the paper [16].
Suppose that $U$ is a separable Banach space with the dual space $U^\star$. For $x \in U$ and $x^\star \in U^\star$ we denote by $(x, x^\star)_{U,U^\star}$ the canonical duality. A $U$-valued Wiener process $W$ is defined as an adapted, Gaussian and continuous process, such that $W(0) = 0$, and such that for every $x^\star \in U^\star$ with $|x|_{U^\star} = 1$ the real valued process $(W(t), x^\star)_{U,U^\star}$ is a Wiener process. Replacing if necessary $U$ by its closed subspace we can assume that $U$ is precisely the support of the law $\mathcal{L}(W(1))$. Then, there exists a unique separable Hilbert space $\mathcal{H}$ densely and continuously embedded into $U$ such that

$$\mathbb{E}(W(t), x^\star)_{U,U^*}(W(s), y^\star)_{U,U^*} = (t \wedge s)\langle x^\star, y^\star\rangle_{\mathcal{H}} \quad \text{for } t, s \geq 0, x^\star, y^\star \in U^*,$$

where we identify $\mathcal{H}^*$ with $\mathcal{H}$, and then $U^*$ with properly chosen subspace of $\mathcal{H}$. Hence, since $U^*$ is dense in $\mathcal{H}$, for any $t \geq 0$ the mapping

$$U^* \ni x \mapsto (W(t), x)_{U,U^*} \in L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

has the unique continuous extension to $\mathcal{H}$. Let us denote this extension also by $W(t)$. Note that $W$ is a cylindrical Wiener process on $\mathcal{H}$. The space $\mathcal{H}$ is called the *reproducing kernel Hilbert space*, shortly RKHS.
Now let $U$ be a Hilbert space such that the embedding $\mathcal{H} \hookrightarrow U$ is dense and Hilbert Schmidt. We identify $U^*$ with a subspace of $\mathcal{H}$ and denote also by $\langle\cdot,\cdot\rangle$ be the bilinear form on $U^* \times U$. Recall that $\langle x, y\rangle = (x, y)_{\mathcal{H}}$ for $x \in U^*$ and $y \in \mathcal{H}$.

**Theorem 2.3.2.** *Let $U$ be a Hilbert space such that the embedding $\mathcal{H} \hookrightarrow U$ is dense and Hilbert Schmidt. Then the following holds.*

*(i) If $W$ is a cylindrical Wiener process on $\mathcal{H}$ then there exists a $U$-valued $Q$-Wiener process $W_Q$ such that*

$$\langle x, W_Q(t)\rangle = W(t,x), \quad t \geq 0, x \in U^*. \tag{2.7}$$

*Moreover, the RKHS of $W_Q$ is equal to $\mathcal{H}$.*

- *Conversely, if $W$ is a Wiener process in $U$ with RKHS equal to $\mathcal{H}$ then (2.7) defines a cylindrical Wiener process on $\mathcal{H}$.*

Let $\mathcal{H}$ be real separable Hilbert space, let $(e_k)$ be an orthonormal basis of $\mathcal{H}$, and let $(\beta_k)$ be a system of independent Gaussian real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Let $U$ be a real Banach space. A bounded linear operator $K : \mathcal{H} \to U$ is said to be $\gamma$-radonifying, or simply radonifying, iff the series $\sum_k \beta_k K e_k$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$.
The set of all $\gamma$-radonifying operators from $\mathcal{H}$ into $U$ is denoted by $R(\mathcal{H}, U)$. Note that if $K \in R(\mathcal{H}, U)$ then $\sum_k \beta_k K e_k$ is a 0-mean Gaussian $U$-valued random variable, and consequently Fernique's theorem yields that

$$|K|_{R(\mathcal{H},U)} \stackrel{def}{=} \left(\mathbb{E}|\sum_k \beta_k K e_k|_U^2\right)^{1/2} = \left(\int_U |e|_U^2 \gamma_K(de)\right)^{1/2} < \infty,$$

where $\gamma_K$ denotes the law of the $U$-valued random vector $\sum_k \beta_k K e_k$. It is obvious that for any $K \in R(\mathcal{H}, U)$, $|K|_{R(\mathcal{H},U)}$ does not depend on the choice $(e_k)$ and $(\beta_k)$. Moreover, $|\cdot|_{R(\mathcal{H},U)}$ is a norm, and $(R(\mathcal{H}, U), |\cdot|_{R(\mathcal{H},U)})$ is a separable Banach space. See [8]
Assume that $U$ is a separable Hilbert space. Recall that a bounded linear operator $K$ from $\mathcal{H}$ into $U$ is called Hilbert-Schmidt iff

$$|K|_{L_{\mathrm{HS}(\mathcal{H},U)}} \stackrel{def}{=} \left(\sum_{k=1}^{\infty} |K e_k|_U^2\right)^{1/2} < \infty,$$

for any orthonormal basis $(e_k)$ of $\mathcal{H}$. Let us denote by $L_{\mathrm{HS}(\mathcal{H},U)}$ the class of all Hilbert-Schmidt operators from $\mathcal{H}$ into $U$. We remark that $R(\mathcal{H}, U) = L_{\mathrm{HS}(\mathcal{H},U)}$ and $|\cdot|_{R(\mathcal{H},U)} = |\cdot|_{L_{\mathrm{HS}(\mathcal{H},U)}}$ [16].
Assume that $W$ is a cylindrical Wiener process on $\mathcal{H}$. Let $(e_k)$ be an orthonormal basis of $\mathcal{H}$. Let $W_k(t) = W(t, e_k)$. Then $(W_k)$ is a sequence of independent standard real valued Wiener process. Let $U$ be a Hilbert space such that the embedding $\mathcal{H} \hookrightarrow U$ is Hilbert Schmidt. Then

$$W(t) = \sum_k W_k(t) e_k \quad t \geq 0,$$

is well defined since the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}, U)$. Clearly, $W$ is a Wiener process on $U$ with the RKHS $\mathcal{H}$, and $W$ is independent on the choice of $e_k$.

### 2.3.4 Hilbert space valued Lévy processes

**Definition 2.3.3** (Lévy process)**.** An $H$-valued Lévy process is a stochastic process $X = \{X(t), t \in [0, \infty)\}$ such that

- $X(0) = 0$ a.s. and $X$ is stochastically continuous: $\forall\ \varepsilon > 0$,
$$\lim_{t \downarrow 0} \mathbb{P}(|X(t)| > \varepsilon) = 0.$$
- $X$ has independent increments, that is, $\forall\ 0 \leq t_0 < t_1 < \cdots < t_n$, the random vectors $X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent,
- $X$ has stationary increments:
$$X_{t+s} - X_t \overset{d}{=} X_s \quad \forall\, s, t \geq 0,$$
- $t \mapsto X(t)$ is càdlàg a.s. .

Note, that without the assumption of stationary increments, we have an *additive process*. We will need also a two-sided Lévy process defined as follows. Let $X_1$ and $X_2$ be two independent Lévy processes defined on the same probability space and with the same distribution. Then we define the two-sided Lévy process

$$X(t) = \begin{cases} X_1(t) & \text{if} \quad t \geq 0 \\ X_2(t) & \text{if} \quad t < 0 \end{cases} \qquad t \in \mathbb{R}\,.$$

For a two-sided Lévy process we will consider the filtration $\mathcal{F}_t = \sigma\,(X(s) : s \leq t)$ for all $t \in \mathbb{R}$.

### 2.3.5 Lévy Khintchine decomposition

A key result in the theory of Hilbert space-valued Lévy processes is the celebrated Lévy-Itô decomposition, in which the sample paths of a given Lévy process are decomposed into continuous and discontinuous parts. More precisely, let $L = (L(t), t \geq 0)$ be an $H$-valued Lévy process. The jump at time $t$ is $\Delta L(t) = L(t) - L(t-)$. For a Borel set $\Gamma \in \mathcal{B}(H \setminus \{0\})$ we define a Poisson random measure

$$\pi_\Gamma(t) := \sum_{0 \leq s \leq t} 1_\Gamma(\Delta L(s)), \quad t \geq 0$$

Note that the definition of Lévy process $L$ implies that $\pi_\Gamma$ is a $Z_+$-valued Lévy process with jump size 1. Moreover, because $L$ has càdlàg paths there are only finitely many jumps of size larger than a positive constant and thus there are only finitely many in the set $\Gamma$. So, $(\pi_\Gamma, t \geq 0)$ is a Possion process with $\mathbb{E}\pi_\Gamma(t) = t\mathbb{E}\pi_\Gamma(1) = t\nu(\Gamma)$. The measure $\nu$ is the so-called Lévy measure which is finite on sets separated from 0. For any Borel set $\Gamma \subset H \setminus \{0\}$, we define

$$L_\Gamma(t) = \sum_{s\le t} 1_\Gamma(\Delta L(s))\Delta L(t)\,.$$

Shortly we will introduce the concept of Poisson random measure on an arbitrary measurable space $(H, \mathcal{H})$ with certain intensity measure. The so-called *compensated Poisson random measure* is defined by

$$\tilde{\pi}_\Gamma(t) := \pi_\Gamma(t) - t\nu(\Gamma),$$

for every $\Gamma \in \mathcal{B}(H\setminus\{0\})$ Finally, we present the following version of Lévy-Khinchin decomposition introduced in [87] which decompose every Lévy process into drift, Brownian, small jump and large jump parts. For he proof see Theorem 4.7 [87] or Theorem 4.1 in [4].

**Theorem 2.3.4.** *If $L = (L(t), t \ge 0)$ is a $H$-valued Lévy process, then the corresponding jump intensity $\nu$ satisfies*

$$\int_H (|y|_H^2 \wedge 1)\nu(dy) < \infty.$$

*Moreover, every Lévy process has the following represtations:*

$$L(t) = bt + W(t) + \sum_{k=1}^\infty \left( L_{\Gamma_k}(t) - t\int_{\Gamma_k} y\nu(dy)\right) + L_{\Gamma_0}(t),$$

*where $\Gamma_0 := \{x : |x|_H \ge r_0\}$, for $k \ge 1$, $\Gamma_k := \{x : r_k \le |x|_H < r_{k-1}\}$, $(r_k)$ is an arbitrary sequence decreasing to 0, $W$ is a Wiener process. Moreover, all members of the representation are independent processes and the series converges $\mathbb{P}$ a.s. uniformly on each bounded subinterval* $[0, \infty)$.

The cylindrical Lévy Noise used in this thesis is obtained by subordinating a cylindrical Wiener process by an arbitrary real valued, increasing Lévy process. This increasing Lévy process is chosen to be a $\frac{\beta}{2}$ stable (symmetric) process, with $\beta \in (0, 2)$. Let us recall some basic facts from [94]. First, recall that a real random variable $X$ is said to be $\beta$-stable with the, scale parameter $\sigma$, skewness parameter $\delta$, and shift parameter $\nu$, shortly $X \sim S_\beta(\sigma, \delta, \nu)$, if

$$\mathbb{E}e^{i\theta X} = e^{i\theta\nu - |\sigma\theta|^\beta(1 - i\delta c\,\mathrm{sgn}(\theta))},$$

where

$$c = \begin{cases} \left((\sigma\theta)^{1-\beta} - 1\right)\tan\frac{\pi\beta}{2} & \text{if} \quad \beta \ne 1 \\ -\frac{2}{\pi}\log|\sigma\theta| & \text{if} \quad \beta = 1 \end{cases}$$

Note that in particular, $S_2(\sigma, 0, \nu) = N(\nu, 2\sigma^2)$ is Gaussian.
We have also the following definition.

**Definition 2.3.5.** A real valued r.v. $X$ is said to be symmetric $\beta$-stable, $0 < \beta \leq 2$, if $X \sim S_\beta(\sigma, 0, 0)$ or, explicitly

$$\mathbb{E}e^{i\theta X} = e^{-\sigma^\beta |\theta|^\beta /2}, \quad \theta \in \mathbb{R}. \tag{2.8}$$

The name "$\beta$-stable" means that if $X_1, \cdots, X_m$ are independent and $\beta$-stable, then $\sum_{j\leq m} \alpha_j X_j$ is $\beta$-stable, and

$$\sigma(\sum_{j\leq m} \alpha_j X_j) = \left( \sum_{j\leq m} |\alpha_j|^\beta \sigma(X_j)^p \right)^{1/\beta},$$

which is obvious from (2.8).

**Definition 2.3.6.** A random vector $X = (X_1, \cdots, X_N)$ with values in $\mathbb{R}^N$ is $\beta$-stable if each linear combination $\sum_{i=1}^N \alpha_i X_i$ is a real $\beta$-stable variable.
A random process $X = (X_t, t \in I)$ indexed by $I$ is called $\beta$-stable if for every $t_1, \cdots, t_N$ in $T$, $(X_{t_1}, \cdots, X_{t_N})$ is a $\beta$-stable random vector. (p.131 in [70], p.233 in [101])
A natural generalisation of the $\mathbb{R}^n$ definition of stable Lévy motion (see for instance p.113 [94]) to the Hilbert space is the following

**Lemma 2.3.7.** *A Lévy process $\{X(t), t \geq 0\}$ on a Hilbert space is a $\beta$-stable Lévy motion if and only if $X(t) - X(s) \sim S_\beta((t-s)^{1/\beta}, \delta, 0)$ for some $0 < \beta \leq 2$, $-1 \leq \delta \leq 1$, $0 < \sigma < \infty$.*

### 2.3.6 Poisson process and Poisson random measure

A Lévy process with values $\mathbb{Z}_+ = \{0, 1, \cdots\}$,, which is increasing with jumps of size 1, is said to be a *Poisson process*. One constructs a Poisson process using exponentially distributed random variables, see chapter 4 in [87].
For all $t > 0$ and $\Gamma \in \mathcal{B}(H\backslash\{0\})$ with $0 \notin \bar{\Gamma}$, we define the Poisson random measure corresponding to $L$ by the formula

$$\pi([0,t], \Gamma) := \pi_\Gamma(t)$$

The process

$$\tilde{\pi}([0,t], \Gamma) := \pi([0,t], \Gamma) - t\nu(\Gamma), \quad t \geq 0, \Gamma \in \mathcal{B}(H\backslash\{0\})$$

is called the compensated Poisson random measure.
Let us now define Poisson random measure on general measure space.

Let

$$\overline{Z}_+ = Z_+ \cup \{+\infty\} = \{0, 1, 2, \cdots\} \cup \{+\infty\}$$

Let $(\Theta, \mathcal{B}, \rho)$ be a $\sigma$-finite measure space. A family of $\overline{Z}_+$-valued r.v. $\{\pi(\Gamma) : \Gamma \in \mathcal{B}\}$ is called a Poisson random measure on $\Omega$ with intensity measure $\rho$, if

- For any measurable set $\Gamma$, the random variable $\pi(\Gamma) \sim$ Poisson$(\rho(\Gamma))$,
- If $\Gamma_1 \cap \Gamma_2 \cap \cdots \cap \Gamma_n = \emptyset$, then $\pi(\Gamma_1), \cdots \pi(\Gamma_n)$ are independent,
- For every $\omega$, $\pi(\cdot, \omega)$ is a measure on $H \setminus \{0\}$.

## 2.4 Poisson random measure and stochastic integration

In this section, we first recall some facts from the theory of Poisson stochastic integration(see [89]). Then we introduce a weaker concept of measurability and present a new version of stochastic Fubini theorem.

**Definition 2.4.1.** Let $\{\tau_i\}_{i\geq 1}$ be a sequence of independent exponential random variables[3] with parameter $\lambda$ and $T_n = \sum_{k=1}^n \tau_k$. The process $(N_t, t \geq 0)$ defined by

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}$$

is called a Poisson process with intensity $\lambda$.
Now Let $\xi_i$ be a sequence of i.i.d. random variables. The process

$$Y(t) = I_{\{N_t \geq 1\}} \sum_{i=1}^{N_t} \xi_i$$

is said to be a *compound Poisson process*. The trajectory of a compound Poisson process is a piecewise continuous function with discontinuities (jumps) at random times (jump times). In other words, a compound Poisson process generates a sequence of pair $(\tau_k, \xi_k)_{k\in\mathbb{N}}$ of *jump times* $\tau_k$ and *marks* $\xi_k$. The size of the jumps is determined by the *marks* $\xi_i$ and the number of jumps up to a time $t$ is determined by $N_t$. We can associate a *random measure* to any counting process as follows. For any Borel set $B \subset \mathbb{R}^+$, for any $\omega$, set

$$\pi(\omega, B) = \#\{k \geq 1 : \tau_k(\omega) \in B\},$$

where $\tau_k$ is the sequence of jump times. The map $B \mapsto \pi(\omega, B)$ defines a positive measure on $\mathbb{R}^+$. One may view the random measure as some form of derivative of a Poisson process. To visualise this, recall that each trajectory $t \mapsto N_t(\omega)$ of a Poisson process is an increasing step

[3]A positive random variable with parameter $\lambda > 0$ is said to be *exponential* if the it has a probability density function of the form $\lambda e^{-\lambda y} 1_{y\geq 0}$.

function. Hence its derivative (in the sense of distributions) is a positive measure: in fact it is simply a superposition of Dirac mass centred at jump times:

$$\frac{d}{dt}N_t(\omega) = \pi(\omega, [0, t]) \quad \text{where} \quad \pi = \sum_{i\geq 1} \delta_{\tau_i}(\omega).$$

$\pi(\omega, dt) = \sum_k \delta_{\tau_k(\omega)}(dt)$. Moreover, the r.v. $N_t$ can be written as

$$N_t(\omega) = \pi(\omega, (0, t]) = \int_{(0,t]} \pi(\omega, ds)$$

and the Stieltjes integral as $\int_0^t f(s)dN(s) = \int_0^t f(s)\pi(ds)$.
Let $H_0 = H \setminus \{0\}$ and let $\lambda$ be a measure on $\mathcal{B}(H_0)$.[4] Consider now on $H_0 \times [0, \infty)$ a given intensity measure of the form

$$\lambda_\rho(dy \times dt) = \rho(dy)dt$$

where $\rho$ is a measure on $H_0$ with

$$\int_{H_0} \min(1, |y|^2)\rho(dy) < \infty$$

The corresponding Poisson random measure $\pi_\rho(\cdot)$ on $H_0 \times [0, \infty)$ is assumed to be a Poisson distributed r.v. with intensity

$$\lambda_\rho(A) = \int_0^T \int_{H_0} 1_{(y,t)\in A}\, \rho(dy)dt$$

counts the number of points in $A \subseteq H_0 \times [0, \infty)$ for $T \in (0, \infty)$ and each set $A$ from the product sigma algebra $\mathcal{B}(H_0)$ and $\mathcal{B}([0, T])$. This implies,

$$\mathbb{P}(\pi_\rho(A) = k) = e^{-\lambda_\rho(A)} \frac{\lambda_\rho(A)^k}{k!}$$

for $k \in \{0, 1, \cdots\}$. For disjoint measurable sets $A_1, \cdots, A_r, \subseteq H_0 \times [0, T]$, the r.v. $\pi_\rho(A_1), \cdots, \pi_\rho(A_r)$, $r \in \mathbb{N}$ are assumed to be independent.

**Definition 2.4.2.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, Hilbert space $H$ and $\lambda$ a given (positive) radon measure $\lambda$ on $(H_0, \mathcal{B}(H_0))$. A Poisson measure on $H_0$ with intensity measure $\lambda$ is an integer valued random measure:

$$\begin{aligned} \pi :& \Omega \times H_0 \to \mathbb{N} \\ &(\omega, A) \mapsto \pi(\omega, A) \end{aligned}$$

such that

[4] We denote by $\mathcal{B}(\Gamma)$ that the smallest sigma-algebra containing all open sets of a set $\Gamma$.

- For (almost all) $\omega \in \Omega$, $\pi(\omega, \cdot)$ is an integer-valued Borel probability measure on $H_0$: For any bounded measurable $A \subset H_0$, $\pi(A) < \infty$ is an integer valued r.v.
- For each measurable set $A \subset H_0$, $\pi(\cdot, A) = \pi(A)$ is a Poisson random measure with parameter $\lambda(A)$ :

$$\mathbb{P}(\pi_\rho(A) = k) = e^{-\lambda(A)} \frac{\lambda(A)^k}{k!}, \quad \forall k \in \mathbb{N}.$$

- For disjoint measurable sets $A_1, \cdots, A_r, \subseteq U_0 \times [0, T]$, the r.v. $\pi_\rho(A_1), \cdots, \pi_\rho(A_r)$, $r \in \mathbb{N}$ are assumed to be independent.

A Poisson random measure on $[0, T] \times H_0$ can be represented as a counting measure:

$$\pi(\omega, \cdot) = \sum_{k \geq 1} \delta_{\tau_k(\omega), \xi_k(\omega)}.$$

Moreover, the integral $I_{f,\pi_\rho}(t)$ for the integrand $f = \{f(t, y), t \geq 0, y \in H_0\}$ is obtained as

$$I_{f,\pi_\rho} = \int_0^T \int_{H_0} f(s, y) \pi_\rho(dy, ds).$$

Let us now introduce a weaker concept of measurability. In particular, Let us first define what is meant by a strongly measurable function.

**Definition 2.4.3.** Let $U$ and $E$ be two separable Hilbert spaces. (1) A function $\Phi : [0, T] \times [0, T] \times \Omega \to L(U, E)$ is said to be simple if there exist integers $N_1, N_2 \geq 1$, partitions $(s_i)$ and $(t_k)$ of $[0, T]$, family of pairwise disjoint $\mathcal{F}_{t_k}$-measurable sets $\{A_{kj} : k \leq N_2 - 1, j \leq N_3\}$ and a family of operators $\{\phi_{ijk} : i \leq N_1, k \leq N_2, j \leq N_3\} \subset L(U, E)$ such that

$$\Phi(s, t, \omega) = \sum_{i=0}^{N_1 - 1} \sum_{k=0}^{N_2 - 1} \sum_{j=0}^{N_3} 1_{(s_i, s_{i+1}]}(s) 1_{(t_k, t_{k+1}]}(t) 1_{A_{kj}}(\omega) \varphi_{ijk} .$$

Let $S(T, T, U, E)$ denote the class of all simple processes.
(2) A function $\Phi : [0, T] \times [0, T] \times \Omega \to L(U, E)$ is said to be strongly measurable if there exists a sequence $(\Phi_n)$ of simple functions, such that for each $y \in U$,

$$|\Phi y - \Phi_n y|_E \to 0, \quad \mathbb{P} - a.s. \quad .$$

Denote by $\mathcal{H}(T, T, U)$ the linear space of all equivalence classes of mapping $\Phi : [0, T] \times [0, T] \times \Omega \to L(U, E)$ strongly $\mathcal{P}_{[0,T] \times [0,T]}$ measurable for which

$$\int_0^T \int_0^T \int_U |\Phi(s, \sigma) y| \pi(dy, d\sigma) ds < \infty, \quad \mathbb{P} - a.s.$$

and such that

- For every $y \in U$ the $E$-valued process $\Phi y$ is predictable

- and

$$\mathbb{P}\left(\int_0^T \int_0^T \int_U |\Phi(s,\sigma)y| \pi(dy, d\sigma)\, ds < \infty\right) = 1.$$

In a similar way we can define a real separable Hilbert space $H(T, T, U)$ endowed with the norm

$$|\Phi|_{H(T,T,U)} := \left(\mathbb{E}\int_0^T \int_0^T \int_U |\Phi(s,\sigma)y|^2 \pi(dy\;, d\sigma)\, ds\right)^{1/2} < \infty$$

is a real separable Hilbert space. It follows that the space $S(T, T, U, E)$ is dense in $H(T, T, U, E)$. A by-product of stochastic integrals w.r.t. Poisson random measures and compensated Poisson random measures is the celebrated Lévy Itô decomposition we have seen earlier, which gives a representation of the non-Gaussian part of a Lévy process via the integrals.
Let $f : [0, T] \to L(U, E)$, and we define the stochastic integral

$$\int_0^T f(s) dL(s).$$

To this end, we follow the idea in [23]. By the Lévy Itô decomposition, any jump Lévy process $L(t)$ can be decomposed into a small jumps[5] process and a large jump process:

$$L(t) = L_1(t) + L_2(t), \quad t \geq 0.$$

Let $\nu$ denotes the intensity measure. Then $L_1$ is the Lévy process corresponding to $\nu_1$ defined by

$$\nu_1(\Gamma) := \nu(\Gamma \cap B_U(0, 1)),$$

where $B_U(0,1) := u \leq U : |u|_U \leq 1.$ the closed unit ball on $U$. The process can be easily constructed in term of Poisson random measure $\pi$ associated with the process $L$ defined by the formula

$$\#\{s \leq t : \Delta L(s) \in \Gamma\} = \pi([0, t] \times \Gamma) := \lim_{\varepsilon \downarrow 0} \sum_{\varepsilon \leq s \leq t} I_\Gamma(\Delta L(s)), \quad \Gamma \in \mathcal{B}(U \backslash \{0\}),$$

where $\Delta L(s) := L(s) - L(s-), s \geq 0$. It can be shown that $\pi$ is a time homogeneous Poisson random measure and $L$ can be expresses in term of the random measure $\pi$ as

$$L(t) = \sum_{s \leq t} \Delta L(s) = \int_0^t \int_U y\pi(dy, ds), \quad t \geq 0.$$

[5]Without loss of generality, assume jump size 1 as cutoff.

Then one can define $L_1$ and $L_2$ by the following slight modification of the above formula.

$$L_2(t) = \sum_{s\in[0,t]} \Delta L(s) I\{\Delta L(s) \geq 1\} = \int_0^t \int_{|y|\geq 1} y\pi(dy, ds),$$

$$L_1(t) = \sum_{s\in[0,t]} \Delta L(s) I\{\Delta L(s) < 1\} = \int_0^t \int_{|y|<1} y\pi(dy, ds).$$

Thus,

$$\int_0^T f(s)dL(s) = \int_0^T f(s)dL_1(s) + \int_0^T f(s)dL_2(s).$$

One can check that both integrals w.r.t. $L_1$ and $L_2$ take values in $E$. (See p.165 [23] for more detail)

We are now ready to present a new version of stochastic Fubini theorem which capture both large and small jumps.

**Theorem 2.4.4.** *Let $U$ and $E$ be separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $T > 0$ be fixed. Assume that the mapping $[0,T] \times [0,T] \times \Omega \ni (s, \sigma, \omega) \mapsto \Phi(s, \sigma, \omega) \in L(U, E)$ is a strongly measurable with respect to the σ-algebra $\mathcal{B}([0,T]) \otimes \mathcal{P}_T$, where $\mathcal{P}_T$ stands for the predictable σ-algebra in $[0,T] \times \Omega$. More precisely, we assume that for every $y \in U$ the mapping $[0,T] \times [0,T] \times \Omega \ni (s, \sigma, \omega) \mapsto \Phi(s, \sigma, \omega)y \in E$ is measurable with respect to the σ-algebra $\mathcal{B}([0,T]) \otimes \mathcal{P}_T$. Furthermore, assume that L is a $U$-valued Lévy process defined as $L(t) := W(Z(t))$, $Z(t)$ is a subordinator process belonging to Sub(p), i.e. $Z(t)$ has intensity measure satisfying*

$$\rho(\{0\}) = 0, \quad \int_1^\infty \rho(d\xi) + \int_0^1 \xi\rho(d\xi) < \infty \quad \int_0^1 \xi^{\frac{p}{2}}\rho(d\xi) < \infty, \tag{3}$$

*where $\rho$ and $\nu$ are respectively the intensity measure on $\mathbb{R}$ and Lévy measure on $U_0$. One relates $\rho$ and $\nu$ as*

$$\nu(\Gamma) = \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \Gamma \in \mathcal{B}(Y).$$

*Then*

$$\int_0^T \left( \int_0^s \Phi(s, \sigma)dL(\sigma) \right) ds = \int_0^T \left( \int_\sigma^T \Phi(s, \sigma)ds \right) dL(\sigma). \tag{4}$$

**Proof.** Assume first that $\Phi$ is a simple process of the form

$$\Phi(s, \sigma, \omega) = I_{(s_1,s_2]}(s) I_{(t_1,t_2]}(\sigma) I_A(\omega)\phi$$

with a bounded operator $\phi : U \to E$ and $A \in \mathcal{F}_{t_1}$. It is easy to check that the theorem holds in this case and by linearity it holds for every $\Phi \in S(T, T, U, E)$. It is also easy to see that the theorem holds for the case when $L = L_2$, that is when $|\Delta L(t)| \geq 1$. Indeed let $\tau_k$, denote jumps of the process $L$. Then only finite number of jumps can occur before time $T$ and therefore

$$\begin{aligned}\int_0^T \left(\int_0^s \Phi(s,\sigma)dL(\sigma)\right) ds &= \int_0^T \sum_{\tau_k \leq s} \Phi(s,\tau_k-)(L(\tau_k) - L(\tau_k-))\, ds \\ &= \sum_{\tau_k \leq T} \left(\int_{\tau_k}^T \Phi(s,\tau_k-)\, ds\right) (L(\tau_k) - L(\tau_k-)). \\ &= \int_0^T \left(\int_\sigma^T \Phi(s,\sigma)ds\right) dL(\sigma)\end{aligned}$$

For the small jump, the proof follows the same lines as p.14 in [3]. □

## 2.5 Subordinator and Subordinated processes in the Hilbert space

A *subordinator* is a real-valued Lévy process which takes nonnegative values only. Combine with the definition of Lévy processes, we will see in the characterization theorem 2.5.2 that subordinators must be an increasing process. Put in another way, *subordinator* is precisely a convolution semigroup $(\mu_t, t \geq 0)$ of probability measures on $\mathbb{R}$ wherein each supp$(\mu_t) \subseteq [0,\infty)$[6]. By increasing we mean that the trajectories of $Z$ are *a.s. nondecreasing*: $Z(t) \geq Z(s)$ whenever $t \geq s$.
Using subordinators, one can construct new and interesting examples of convolution semigroups.
Suppose $\lambda$ is a measure on $\mathbb{R}$ then we denote its Laplace transform by

$$\tilde{\lambda}(r) = \int_{\mathbb{R}} e^{-r\xi}\lambda(d\xi)$$

for all values of $r$ such that the integral remains finite. In particular, if $\lambda$ has support on $[0,\infty)$ then $\tilde{\lambda}(r)$ is defined (perhaps infinite) at least for $r \geq 0$.
Using the Lévy Itô decomposition, one can show that any subordinator must have its diffusion coefficient as zero, drift $b$ must be nonnegative, and the Lévy measure $\rho$ cannot be on $(-\infty,0)$. More precisely, one has the following fundamental characterization theorem of subordinator processes (See Theorem 2.1 in [23], also Theorem 21.5 [95]).

**Lemma 2.5.1** (p.53[87])**.** *For every Borel set $\Gamma$, such that the closure of $\Gamma$ does not contain 0 and for all $r \geq 0$,*

$$\mathbb{E}e^{rZ(t)} = \exp\left(t\int_\Gamma (1 - e^{rx})\right) \nu(dx).$$

**Proof.** By an easy limiting argument one can assume that $\Gamma$ is bounded. Given $\delta > 0$, let $\Gamma_1, \cdots, \Gamma_m$ be disjoint sets of diameters less than $\delta$ and such that $\Gamma = \cup_{k=1}^m \Gamma_k$. In addition let $x_k \in \Gamma_k$, $k = 1, \cdots, m$. Then

$$\begin{aligned}\left| Z_\Gamma(t) - \sum_{k=1}^m x_k \pi_{\Gamma_k}(t) \right|_H &\leq \sum_{k=1}^m |Z_\Gamma(t) - x_k \pi_{\Gamma_k}(t)|_H \\ &\leq \delta \sum_{k=1}^m \pi_{\Gamma_k}(t) = \delta \pi_\Gamma(t),\end{aligned}$$

and so

$$\sum_{k=1}^m x_k \pi_{\Gamma_k}(t) \to Z_\Gamma(t)$$

$\mathbb{P}$-a.s., as $\delta \to 0$. Consequently,

$$\begin{aligned}\mathbb{E}e^{rZ_\Gamma(t)} &= \lim_{\delta\to 0} \mathbb{E} \exp \left( r \sum_{k=1}^m x_k \pi_{\Gamma_k}(t) \right) \\ &= \lim_{\delta\to 0} \prod_{k=1}^m \mathbb{E} \exp \left( (r x_k) \pi_{\Gamma_k} \right) \\ &= \lim_{\delta\to 0} \prod_{k=1}^m \exp \left( t \nu(\Gamma_k)(1 - e^{r x_k}) \right) \\ &= \exp \left( t \int_\Gamma (1 - e^{r x_k}) \nu(dx) \right).\end{aligned}$$

□

**Theorem 2.5.2.** *Suppose that $Z = (Z(t), t \geq 0)$ is a subordinator process defined on the probability space $\mathfrak{B} = (\Omega, \mathcal{F}, \mathbb{P})$, then there exists a real number $b \in \mathbb{R}^+$, a non-negative measure $\rho$ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ satisfying*

$$\rho(\{0\}) = 0, \quad \int_1^\infty \rho(dx) + \int_0^1 x\rho(dx) < \infty, \tag{2.9}$$

*such that*

$$\mathbb{E}e^{-rZ(t)} = e^{-t\psi(r)}, \quad r \geq 0, t \geq 0, \tag{2.10}$$

*where*

$$\psi(r) = br + \int_0^\infty (1 - e^{-rx})\rho(dx), \ r \geq 0. \tag{2.11}$$

The proof in [23] is missing. For completeness, we fill in the gap with the proof in [95].

**Proof.** Sufficiency. ("$\Longleftarrow$") It follows from $\int_{(-\infty,0)}\rho(dx) = 0$ and the definition of Poisson random measure (Sato [95], Theorem 19.2(i)) that $J_L((0,t]\times(-\infty,0)) = 0$. a.s., meaning that $Z$ does not have any negative jumps. Then, by the Lévy Itô decomposition with the Lévy measure satisfying $\int_{|x|\le 1}|x|\rho(dx) < \infty$ (Sato, Theorem 19.3), $L = L(t)$ takes the form

$$L(t) = L_t^1 + L_t^2 L(t) = \underbrace{\int_{(0,t]\times(0,\infty)} x\pi(d(s,x))}_{L_t^3} + \underbrace{tb}_{L_t^4} \quad \text{a.s.,}$$

where

$$L_t^1 = \int_{(0,t]\times(0,\infty)} x\pi(d(s,x)), \quad L_t^2 = tb.$$

From the fact $L_t^2 = tb$ we see that $L_t(\omega)$ is clearly an increasing function of $t$. This shows $L_t$ is increasing.

Necessity. ("$\Longrightarrow$") Since $Z_t$ has no negative jumps, we know from the definition of random measure (Sato [95], theorem 19.2(i) ) that $\rho((-\infty,0)) = 0$. Since an increasing function stays increasing after finite number of jumps deleted. Hence, the limit for the sum of jumps $L_\varepsilon(t)$, denoted as $\tilde{L}(t)$ exists, so

$$\tilde{L}(t) = \lim_{\varepsilon\downarrow 0} L_\varepsilon(t) = \int_{(0,t]\times(0,\infty)} x\pi(d(s,x),\omega),$$

and this limit is bounded above by $L(t)$, so $\tilde{L}(t) \le L(t)$. Then using Proposition 2.5.1, one has

$$\begin{aligned}\mathbb{E}e^{-rL_\varepsilon(t)} &= \exp\left\{t\int_{(\varepsilon,\infty)}(e^{-rx}-1)\rho(dx)\right\}\\ &= \exp\left\{t\int_{(\varepsilon,\infty)}(e^{-rx}-1+rx\mathbf{I}_{(0,1]}(x))\rho(dx) - tr\int_{(\varepsilon,1]}x\rho(dx)\right\}.\end{aligned}$$

As $\varepsilon\downarrow 0$, on the left hand side we have

$$\mathbb{E}e^{-rL_\varepsilon(t)} \to \mathbb{E}e^{-r\tilde{L}(t)} \ge 0, \quad \text{for } u > 0.$$

Where the exponent $-r\tilde{L} < 0$ and so $e^{-r\tilde{L}} < 1$, the convergence of left hand side is clear, by dominated convergence. Notice on the right hand side that

$$\int_{(\varepsilon,\infty)}(e^{-rx}-1+rx\mathbf{I}_{(0,1]}(x))\nu(dx) \longrightarrow \int_{(0,\infty)}(e^{-rx}-1+rx\mathbf{I}_{(0,1]}(x))\nu(dx) \quad \text{as} \quad \varepsilon\downarrow 0,$$

which is finite as

$$\int_H (1\wedge x^2)\nu(dx) < \infty$$

Hence,

$$\int_{[0,1]} x\rho(dx) < \infty$$

Now by the Lévy Ito decomposition, one has

$$L_t = \underbrace{L_t^1}_{jump} + L_t^2,$$

which implies,

$$L_t^3 = \tilde{L}(t)$$

and $L_t^4$ has triple $(b, Q, 0)$. But

$$L_t^4 = L_t - \tilde{L}(t) \geq 0$$

implies

$$Q = 0 \quad \text{and} \quad b \geq 0.$$

□

**Definition 2.5.3** (p.156,[23])**.** For $p \geq 0$, denote by Sub$(p)$ the set of all subordinator process $Z$ whose intensity measure $\rho$ satisfies

$$\int_0^1 \xi^{\frac{p}{2}} \rho(d\xi) < \infty. \tag{2.12}$$

**Example 1.** (1) It is obvious that if

$$0 < p_1 < p_2 < 2 \leq p_3,$$

then

$$\text{Sub}(p_1) \nsubseteq \text{Sub}(p_2) \nsubseteq \text{Sub}(2) = \text{Sub}(p_3).$$

(2) Note, if $\beta \in (0, 1)$ and the measure $\rho$ is defined by

$$\rho(d\xi) = \frac{1}{\beta\Gamma(1-\beta)\xi^{1+\beta}} 1_{(0,\infty)}(\xi) d\xi,$$

where $\Gamma$ is the Euler-gamma function ($\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, Re$z > 0$).
(3) Note that $z^\beta \in$ Sub$(p)$ iff $p > 2\beta$, i.e. $\beta < \frac{p}{2}$. In particular, $Z^\beta \in$ Sub$(1)$ iff $\beta < \frac{1}{2}$.
(4) A standard Poisson process $N$ with rate 1 is a subordinated process with drift $b = 0$ and the intensity measure $\rho = \delta_1$. Here, $\delta_1$ denotes the Dirac Delta function at 1.

Now Let us examine a couple of examples of convolution semigroups. In essence, they are the consequences of the Lévy Itô decomposition where $X$ is (a) a subordinator, (b) $\beta$-stable process, (c) a subordinated process.

**Proof.** For (a), it is clear in view of Theorem 2.5.2. For (b), in the case $\beta \in (0,1)$, we are in the framework of Theorem 2.5.2. Assuming that $b = 0$ and $\rho(d\xi) = \xi^{-1-\beta}d\xi$ for $\beta \in (0,1)$. Then

$$\psi(r) = \int_0^\infty (1 - e^{-r\xi}) \frac{d\xi}{\xi^{1+\beta}} \quad r > 0.$$

For $r > 0$, the change of variable $\eta = r\xi$ gives

$$\psi(r) = \int_0^\infty (1 - e^{-r\xi}) \frac{d\xi}{\xi^{1+\beta}} = \int_0^\infty (1 - e^{-\eta}) \frac{r^\beta d\eta}{\eta^{1+\beta}} = r^\beta \psi(1).$$

Integrating by parts, one obtain $\psi(1) = \frac{1}{\beta\Gamma(1-\beta)}$. For the case $\beta \in (1,2)$, assume that $\beta \in (1,2)$ and

$$\psi(r) = \int_0^\infty (1 - e^{-r\xi} - r\xi) \frac{d\xi}{\xi^{1+\beta}}, \quad r > 0.$$

Then, in the same manner as in the previous case, one obtains that $\psi(r) = r^\beta \psi(1)$, $r > 0$ where $\psi(1) = -\frac{1}{\beta(\beta-1)\Gamma(2-\beta)}$. The proof of (c) will be cleared after introducing Theorem 2.5.4. $\square$

One useful application of subordinators is to produce new convolution semigroups of measures on Hilbert spaces. Namely, suppose $(\zeta_t, t \geq 0)$ is a convolution semigroup of probability measures on a Hilbert space $H$ with exponent $\lambda$, that is, $\int_H e^{i\langle x,y\rangle_H}\zeta_t(dy) = e^{-t\lambda(x)}$. Let $(\mu_t)$ be a convolution semigroup of probability measures on $[0,\infty)$ such that $\int_0^\infty e^{-r\xi}\mu_t(d\xi) = e^{-t\psi(r)}$. The subordinated law $(\tilde{\zeta}_t)_{t\geq 0}$ on $H$ is defined by

$$\tilde{\zeta}_t := \int_0^\infty \zeta_s \mu_t(ds), \quad t \geq 0$$

is a convolution semigroup of measures with exponent $\tilde{\lambda}(x) = \psi(\lambda(x))$, $x \in H$.
**Claim** : $(\tilde{\zeta}_t, t \geq 0)$ is a convolution semigroup on $H$ and

$$\tilde{\zeta}_{t+s} = \tilde{\zeta}_t * \tilde{\zeta}_s, \quad \forall \quad t, s \geq 0.$$

The proof is missing in [23] , we fill in this gap here. The key is to verify the three properties in the definition of convolution semigroup. Namely,

(i) $\tilde{\zeta}_0 = \delta_0$
(ii) $\tilde{\zeta}_t \to \tilde{\zeta}_0$ weakly as $t \downarrow 0$,
(iii) $\tilde{\zeta}_t * \tilde{\zeta}_s = \tilde{\zeta}_{t+s}$, $t, s \geq 0$

**Proof.** For (i) we have

$$\begin{aligned}\tilde{\zeta}_0 &= \int_0^\infty (\zeta_s)\mu_0(ds) \\ &= \int_0^\infty (\zeta_s)\delta_0(ds) \\ &= \zeta_0 = \delta_0\end{aligned}$$

by definition of $\zeta$. To prove (ii) it is enough to show that for every $r > 0$

$$\lim_{t\to 0} \tilde{\zeta}_t(B_r^c) = 0,$$

where $B_r^c$ is the complement of a centered ball of radius $r$ in $H$. To this end we note that

$$\tilde{\zeta}_t(B_r^c) = \int_0^\infty \zeta_s(B_r^c)\mu_t(ds)$$

and since $\mu_t$ and $\zeta_s$ are convolution semigroups, for every $\epsilon > 0$ and $a > 0$ we can find $t_0 > 0$ such that $\mu_t([a,\infty) < \epsilon$ for $t < t_0$, and for all $s < t_0$ $\zeta_s(B_r^c) < \epsilon$. Therefore, for $t < t_0$

$$\tilde{\zeta}_t(B_r^c) \leq \int_0^a \zeta_s(B_r^c)\mu_t(ds) + \epsilon \leq 2\epsilon.$$

Finally, to check *Semigroup property*, Let $(\zeta_s, s \geq 0)$ be a convolution semigroup on $H$. Let $(\mu_s, s \geq 0)$ be a subordinator on $\mathbb{R}^+$. By the Riesz representation theorem , for each $t \geq 0$, there exists a probability measure $\tilde{\zeta}_t$ on $U$ such that for any $f \in C_c(U)$,

$$I_t(f) = \int_U f(\sigma)\tilde{\zeta}_t(d\sigma) = \int_{(0,\infty)} \int_U f(\sigma)\zeta_s(d\sigma)\mu_t(ds).$$

The relationship between the 3 families of measures is frequently expressed using the *vague integral*

$$\tilde{\zeta}_t(A) := \int_0^\infty \zeta_s(A)\mu_t(ds) \quad \forall \quad t \geq 0, A \in \mathcal{B}(U). \tag{5.7.21}$$

It is clear that the subordinated law form a convolution semigroup as well, namely, $(\tilde{\zeta}_t, t \geq 0)$ is a convolution semigroup of measures on $U$ and so $\tilde{\zeta}_{t+s} = \tilde{\zeta}_t * \tilde{\zeta}_s$.
For all $s, t \geq 0$, $f \in C_c(U)$

$$\begin{aligned}\int_U f(x)\tilde{\zeta}_{s+t}(dx) &= \int_U f(x) \int_0^\infty \zeta_r(dx)\mu_{s+t}(dr) \\ &= \int_U \underbrace{\int_0^\infty f(x)\zeta_r(dx)}_{\tilde{f}(r)} \mu_{s+t}(dr) \\ &= \int_U \tilde{f}(r)\mu_{s+t}(dr).\end{aligned}$$

Now by Generalised Fubini theorem, (see Proposition 35.14 [84]), since

$$\int |f(x)\zeta_r(dx)|\mu_{s+t}(x,dr) < \infty \quad \text{a.e.} \quad x(\tilde{\zeta}_{s+t})$$

Then

$$\int_U f(x)\zeta_r(dx)\,d\nu = \int_U f(x)\zeta_r(dx)\int_0^\infty \chi_E(x,r)\mu_{s+t}(x,dr) = \int_U\int_0^\infty f(x)\mu_{s+t}(x,dr)\zeta_r(dx),$$

where $E \in \mathcal{B}_1$.

$$\nu(E) = \int_U\left(\int_0^\infty \chi_E(x,r),\mu_{s+t}(x,r)\right)\zeta_s(dx).$$

Hence, we can swap the integral $\int_U\int_0^\infty\cdots$ to $\int_0^\infty\int_U\cdots$ legitimately now and to continue on,

$$\begin{aligned}
\int_U f(x)\tilde{\zeta}_{s+t}(dx) &= \int_U f(x)\int_0^\infty \zeta_r(dx)\mu_{s+t}(dr)\\
&= \int_U\int_0^\infty \tilde{f}(r)\mu_{s+t}(dr)\\
\int_{(0,\infty)}\int_U f(x)\zeta_r(dx)\mu_{s+t}(dr) &= \int_0^\infty\int_U f(x)\zeta_r(dx)\mu_s(dr-z)\mu_t(dz)
\end{aligned}$$

Now, let $r - z = v, dr = dv$, as $z$ is a constant.

$$\begin{aligned}
&= \int_0^\infty\int_0^\infty\int_U f(x)\zeta_{z+v}(dx)\mu_s(dv)\mu_t(dz)\\
&= \int_0^\infty\int_0^\infty\int_U f(x)\int_U \zeta_z(dx-v)\zeta_v(dv)\mu_s(dv)\mu_t(dz)\\
&= \int_0^\infty\int_0^\infty\int_U\int_U f(x)\zeta_z(dx-v)\zeta_v(dv)\mu_s(dv)\mu_t(dz)\\
&= \int_U\int_U f(x)\int_0^\infty \zeta_z(dx-v)\mu_t(dz)\int_0^\infty \zeta_v(dv)\mu_s(dv)\\
&= \int_U\int_U f(x)\tilde{\zeta}_t(dx-v)\tilde{\zeta}_s(dv)\\
&= \int_U f(x)\int_U \tilde{\zeta}_t(dx-v)\tilde{\zeta}_s(dv)\\
&= \int_U f(x)(\tilde{\zeta}_t * \tilde{\zeta}_s)(dx).
\end{aligned}$$

Whence, $\tilde{\zeta}_{t+s} = \tilde{\zeta}_t * \tilde{\zeta}_s$. □

### 2.5.1 Stable process as a Subordinated cylindrical Wiener process

Lévy processes form a very rich class of processes. However, general Lévy processes are not very tractable. Subordinated cylindrical Wiener processes are obtained from Cylindrical

Wiener processes by replacing its time parameter $t$ by independent subordinators, i.e. increasing Lévy processes starting from 0. In particular, Let $W = (W(t), t \geq 0)$ be a $U$-valued Wiener process, and let $Z = (Z(t), t \geq 0)$ be a independent subordinator $Z^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$, i.e. where $\rho$ is defined by formula (2.12) with $\beta = \frac{\alpha}{2}$. Then the process $(L(t), t \geq 0)$ defined by $L(t) := W(Z(t))$, $t \geq 0$ is called the subordinated $H$-cylindrical Wiener process, or the $H$-cylindrical $\alpha$-stable process. Essentially, we have constructed a stable process with parameter range in $(0, 2)$ via subordinating a $H$-cylindrical Wiener process with a stable process with parameter $\beta \in (0, 1)$. More explicitly, Let $W = (W(t), t \geq 0)$ be a (cylindrial) Brownian motion on $U$ having the form

$$W(t) := \sum_k W_k(t) e_k, \quad t \in [0, T]$$

where $W_k(t) = W(t, e_k)$ is a sequence of independent standard Brownian motion in $\mathbb{R}$ on some $(\Omega, \mathcal{F}, \mathbb{P}; U)$. and $e_k$ is an orthonormal basis of $H$. We remark that this series does not converge in $H$ as dim $H = \infty$. It does converge, however in any Hilbert space $U$ such that the embedding $H \hookrightarrow U$ is Hilbert-Schmidt.

Consider the subordinator $Z^{\frac{\beta}{2}}$, where $\beta \in (0, 2)$, that is, an increasing one dimensional Lévy process with Laplace transform

$$\mathbb{E} e^{-rZ(t)} = e^{-tr^{\frac{\beta}{2}}}, \quad r > 0$$

The subordinated cylindrical Wiener process $(L_t, t \geq 0)$ on $H$ is defined by

$$L(t) := W(Z(t)).$$

We remark that, in general, $L(t)$ do not belong to $H$. Indeed, $L$ lives on some Banach space $U \supset H$ and the embedding $H \hookrightarrow U$ is $\gamma$ radonifying.

Let $W$ be a Wiener process associated with the convolution semigroup $(\zeta_s, s \geq 0)$, Let $Z$ be a Lévy processes associated with the convolution semigroup $(\mu_s, s \geq 0)$, assume $W$ is independent to $Z$. We are now focus our studies in the subordinator defined by

$$L(t) := W(Z(t)).$$

**Theorem 2.5.4** (Theorem 2.4 [23]). *Suppose that $H$ is a separable Hilbert space and $U$ is a separable Banach space such that $H \subset U$ continuously and densely. Assume that $Z$ is a subordinator process with the intensity measure $\rho$ and the drift $b$. Assume that $W = (W(t), t \geq 0)$ is an $U$-valued Wiener process with the Reproducing Kernel Hilbert Space (RKHS) of $W(1)$ equal to $H$. Let us denote $\zeta_s = \mathcal{L}(W(s), s \geq 0)$.*

*i) If the process $L$ is defined by $(L := W(Z(t), t \geq 0))$ then*

$$\mathbb{E}e^{i\langle L(t),\phi\rangle_{U,U^*}} = e^{-t\lambda(\phi)},\ \phi \in U^*, \quad t \geq 0, \tag{2.13}$$

*where the function $\lambda$ is defined by*

$$\lambda(\phi) = \psi\left(\frac{1}{2}|\phi|_H^2\right), \quad \phi \in H, \tag{2.14}$$

*with $\psi$ being defined by (2.3).*

*ii) Moreover $L$ is a $U$-valued Lévy process such that*

$$\mathbb{E}e^{i\langle L(t),\phi\rangle} = e^{-t\int_U(1-e^{i\langle u,\phi\rangle})\nu(du)}, \quad \phi \in U^*,\ t \geq 0 \tag{2.15}$$

*where the measure $\nu$ is given by*

$$\nu(\Gamma) = \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \Gamma \in \mathcal{B}(U), \tag{2.16}$$

*and*

$$PV\int_U (1 - e^{i\langle u,\phi\rangle})\nu(du) := \lim_{\varepsilon\downarrow 0}\int_{u\in U:|u|\geq\varepsilon}(1 - e^{i\langle u,\phi\rangle})\nu(du).$$

*iii) The process $L$ is of finite variation iff*

$$\int_0^1\left(\int_{B_U(0,1)}|u|_U\zeta_s(du)\right)\rho(ds) < \infty.$$

*The process $L$ will be called an $H$-cylindrical Lévy process subordinated by the (subordinator) process $Z$.*

**Proof.** (i) Observe first that the process $L$ is a well defined $U$-valued càdlàg process. For brevity, assume the two independent stochastic processes $W$ and $Z$ are defined respectively on the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ and $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$. Then for any $\phi \in H$, via conditional expectation,

$$\mathbb{E}e^{i\langle L(t),\phi\rangle_{U,U^*}} = \mathbb{E}_1\mathbb{E}_2(e^{i\langle W(z(t,\omega_2),\omega_1),\phi\rangle_{U,U^*}}) \quad \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2$$

Since we want to integrate out $\omega_1$, swap $\mathbb{E}_2$ with $\mathbb{E}_1$

$$= \mathbb{E}_2\mathbb{E}_1(e^{i\langle W(z(t,\omega_2),\omega_1),\phi\rangle_{U,U^*}})$$

Then it follows from infinite divisibility that,

$$= \mathbb{E}_2 e^{-Z(t,\omega_2)\frac{1}{2}|\phi|^2}$$

$$= e^{-t\psi(\frac{1}{2}|\phi|^2)}$$

$$= e^{-t\lambda(\phi)}$$

□

(ii) From (i), we have

$$\mathbb{E}e^{i\langle Y(t),\phi\rangle} = e^{-t\psi(\frac{1}{2}|\phi|^2)}.$$

The question now, is to find Lévy triplet for $L(t)$, who lives in a Hilbert space. The idea is to create a new convolution semigroup of measure on $U$ with exponent $\varphi(\phi) : \int_U e^{i\langle\phi,s}\zeta_t(ds) = e^{-t\varphi(\phi).}$. On the other hand, let $\mu_t$ be a convolution semigroup of measure on $[0,\infty)$ with exponent $\psi(r) : \int_0^\infty e^{-r\xi}\mu_t(d\xi) = e^{-t\psi(r)}$.Then, by a direct computation, the following

$$\zeta_t^{\mu_t}(\Gamma) := \int_0^\infty \zeta_s(\Gamma)\mu_t(ds), \quad t \geq 0$$

is a convolution semigroup of measures with exponent $\lambda(\phi) = \psi(\varphi(\phi))$. To be consistant with the notations used in the paper, replace $\mu_t$ with $\rho$, $\zeta_t$ with $\nu$. We will now see how the Lévy measure

$$\nu(\Gamma) := \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \gamma \in \mathcal{B}(U).$$

**Proof.** (ii)

$$\begin{aligned}
\mathbb{E}e^{i\langle L(t),\phi\rangle} &= e^{-t\psi(\frac{1}{2}|\phi|^2)} \\
&= \exp\left\{-t\int_0^\infty (1 - e^{-\frac{s}{2}|\phi|_H^2})\rho(ds)\right\}
\end{aligned}$$

Note, $\mathbb{E}e^{i\langle L(t),\phi\rangle} = \int_U e^{i\langle\phi,u\rangle}\zeta_s(du) = e^{-\frac{s}{2}|\phi|_H^2}$

$$= \exp\left\{-t\int_0^\infty (1 - \int_U e^{i\langle u,\phi\rangle}\zeta_s(du))\rho(ds)\right\}.$$

The integral $\int_U e^{i\langle u,\phi\rangle}\zeta_s(du)$ may not converge as $\zeta_s(du)$ may not be finite at 0. So, take principle value with cut-off at 0.

$$\begin{aligned}
&= \exp\left\{-t\int_0^\infty \lim_{\varepsilon\downarrow 0}(1 - \int_{u\in U:|u|\geq\varepsilon} e^{i\langle u,\phi\rangle}\zeta_s(du))\rho(ds)\right\} \\
&= \exp\left\{-t\int_0^\infty \mathrm{PV}\int_U (1 - e^{i\langle u,\phi\rangle}\zeta_s(du))\rho(ds)\right\} \\
&= \exp\left\{-t\mathrm{PV}\int_0^\infty \int_U (1 - e^{i\langle u,\phi\rangle}\zeta_s(du))\rho(ds)\right\} \\
&= \exp\left\{-t\mathrm{PV}\int_U (1 - e^{i\langle u,\phi\rangle}\nu(du)\right\}, \quad \phi \in U^*, t \geq 0
\end{aligned}$$

where the measure $\nu$ is given by formula

$$\nu(\Gamma) = \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \Gamma \in \mathcal{B}(U),$$

and

$$\mathrm{PV}\int_U (1 - e^{i\langle u,\phi\rangle}\nu(du) := \lim_{\varepsilon\downarrow 0}\int_{u\in U:|u|>\varepsilon}(1 - e^{i\langle u,\phi\rangle})\nu(du).$$

Moreover, in view of (2.9) and (2.13) and assuming no drift, one has

$$\mathrm{PV}\int_U (1 - e^{i\langle u,\phi\rangle}\nu(du) = \int_0^\infty \mathrm{PV}\int_U (1 - e^{i\langle u,\phi\rangle})\zeta_s(du)\rho(ds).$$

These conclude the proofs of (i) and (ii) □

**Proof.** (iii) Recall, an $U$-valued Lévy process $L$ with intensity measure $\nu$ is of finite variation iff. $\int_{B_U(0,1)} |u|_U \nu(du) < \infty$. Let now $L(t)$ and $\nu$ be as found in (i) and (ii), i.e.

$$L(t) := W(Z(t)) \quad \text{with} \quad \mathbb{E}e^{i\langle L(t),\phi\rangle} = e^{-t\mathrm{PV}\int_U (1-e^{i\langle u,\phi\rangle}\nu(du)} \ \phi \in U^*, t \geq 0,$$

$$\nu(\Gamma) = \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \Gamma \in B(U)$$

Since then

$$\begin{aligned}
\int_{B_U(0,1)} |u|_U \nu(du) &= \int_0^\infty \left(\int_{B_U(0,1)} |u|_U \zeta_s(du)\right)\rho(ds) \\
&= \int_0^1 \left(\int_{B_U(0,1)} |u|_U\right)\rho(ds) + \int_1^\infty \left(\int_{B_U(0,1)} |u|_U \zeta_s(du)\right)\rho(ds) \\
&= \int_0^1 \left(\int_{B_U(0,1)} |u|_U \zeta_s(du)\right)\rho(ds) + \int_1^\infty \rho(ds) \quad (\dagger)
\end{aligned}$$

In view of (2.1), $\int_0^1 \xi\rho(d\xi) + \int_1^\infty \rho(d\xi) < \infty$

$$< \infty \quad \text{iff.} \quad \int_0^1 \left(\int_{B_U(0,1)} |u|_U \zeta_s(du)\right)\rho(ds) < \infty \quad (\ddagger)$$

Well, if (†) exists, and since $\int_1^\infty \rho(ds)$ by (2.1), then necessarily (‡) holds. Conversely, if (‡) holds, it is clear that (†) must be finite. When the subordinated Lévy process has finite variation. □

**Remark.** (1) Since $H$ is assumed to be the RKHS of $W(1)$, this implies the embedding $H \hookrightarrow U$ is $\gamma$-radonifying. (2) In view of Fernique Theorem, $\zeta_s$ is Gaussian and so has finite second moment. Moreover, there exists $C > 0$ such that for all $s > 0$, $\int_U |u|^2\zeta_s(du) \leq Cs$. (3) Given a separable Hilbert space $H$ and a real number $p \in (0, \infty)$ we will denote by $\mathrm{LSub}(H, p)$ that the class of all Lévy processes $L$ of the form $(L(t) := W(Z(t)), t \geq 0)$, where $W$ is $H$-cylindrical Wiener process and $Z$ is an independent subordinator of class $\mathrm{Sub}(p)$. (4) In the special case of the subordinator process $Z^{\frac{\alpha}{2}}$, with $\alpha \in (0, 2)$, that is, when $\rho$ is defined by formula (2.12) with $\beta = \frac{\alpha}{2}$, the process $L$ constructed in view of Theorem 2.5.4 will be denoted by $L^\alpha$ and is called the $H$-cylindrical $\alpha$-stable process. Note that

$$\mathbb{E}e^{i\langle L^\alpha(t),\phi\rangle} = e^{-t\lambda_\alpha(\phi)}, \quad \phi \in U^*, t \geq 0$$

where $\lambda_\alpha$ is defined by

$$\lambda_\alpha(\phi) = \left(\frac{1}{2}\right)^{\frac{\alpha}{2}} |\phi|_H^\alpha, \quad \phi \in H \tag{2.17}$$

(5) It follows from Example 1 (ii) that $L^\alpha$ belongs to the class LSub$(H, p)$ iff $\alpha < p$.
(6)Suppose the Lévy process $L$ and its intensity measure $\nu$ are defined as in Theorem 2.5.4. By part (iii) of the teorem, the process $L$ is of finite variation iff. $\int_{B_U(0,1)} |u|_U \nu(du) < \infty$.
(7) It follows from part (5) of these remarks that, if in addition $H = U = \mathbb{R}$, the process $L^\alpha$ is of finite variation in $U$ iff. $\alpha \in (0, 1)$, i.e.

$$\int_0^1 s^{1/2} s^{-1-\frac{\alpha}{2}} ds < \infty.$$

**Proof.** The proofs of (1)-(3), (5) are trivial. We prove (4), (6),(7). To see why (2.17) is true, we use (2.11) with 0 drift. Then by Theorem (2.5.2),

$$\begin{aligned}\lambda(\phi) &= \psi\left(\frac{1}{2}|\phi|_H^2\right) \\ &= \frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})} \int_0^\infty (1 - e^{-\frac{1}{2}|\phi|_H^2 \xi}) \xi^{-1-\frac{\alpha}{2}} \, d\xi.\end{aligned}$$

Let $a = \frac{1}{2}|\phi|^2\xi$ and $da = \frac{1}{2}|\phi|^2 d\xi$, then

$$
\begin{aligned}
&= \frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\int_0^\infty (1-e^{-a})\left(\frac{a}{\frac{|\phi|^2}{2}}\right)^{-1-\frac{\alpha}{2}}\frac{da}{\frac{|\phi|^2}{2}}\\
&= \frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\int_0^\infty (1-e^{-a})a^{-1-\frac{\alpha}{2}}\left(\frac{2}{|\phi|^2}\right)^{-1-\frac{\alpha}{2}+1} da\\
&= \left(\frac{|\phi|^2}{2}\right)^{\frac{\alpha}{2}}\frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\int_0^s e^{-r}dr\frac{da}{a^{1+\frac{\alpha}{2}}}\\
&= \left(\frac{1}{2}\right)^{\frac{\alpha}{2}}|\phi|_H^\alpha\frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\int_0^s e^{-r}dr\left(\frac{\alpha}{2}\right)e^{-\frac{\alpha}{2}}\\
&= \frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\left(\frac{1}{2}\right)^{\frac{\alpha}{2}}|\phi|_H^\alpha\int_0^\infty e^{-r}\,dr\left(\frac{\alpha}{2}\right)r^{-\frac{\alpha}{2}}\\
&= \frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\left(\frac{1}{2}\right)^{\frac{\alpha}{2}}|\phi|_H^\alpha\frac{\alpha}{2}\int_0^\infty e^{-r}r^{1-\frac{\alpha}{2}-1}dr\\
&= \frac{1}{\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})}\left(\frac{1}{2}\right)^{\frac{\alpha}{2}}|\phi|_H^\alpha\frac{\alpha}{2}\Gamma(1-\frac{\alpha}{2})\\
&= (\frac{1}{2})^{\frac{\alpha}{2}}|\phi|_H^\alpha.
\end{aligned}
$$

Now, Let $L$ be a $U$-valued Lévy process with intensity measure $\nu$ be defined as in Theorem 2.4.

$$\mathbb{E}e^{i\langle L(t),\phi\rangle} = e^{-t\lambda_\alpha(\phi)}, \quad \phi \in U^*, \; t \geq 0,$$

where $\lambda_\alpha(\phi) = \left(\frac{1}{2}\right)^{\frac{\alpha}{2}}|\phi|_H^\alpha$, $\phi \in H$. Then, $L$ is of finite variation iff.

$$
\begin{aligned}
&\int_{B_U(0,1)}|u|_U\nu(du) < \infty\\
\iff &\int_{B_U(0,1)}|u|_U\int_0^\infty \zeta_s(du)\rho(ds) < \infty\\
\iff &\int_0^\infty\left(\int_{B_U(0,1)}|u|_U\zeta_s(du)\right)\rho(ds) < \infty
\end{aligned}
$$

So (4), (6) are proved. Now we prove (7). From (5), we see $L^\alpha \subset LSub(H,p)$ iff. $\alpha < p$. Now let $H = U = \mathbb{R}$, the process $L^\alpha$ is of finite variation in $U$ iff. $\int_0^1 s^{1/2}s^{-1-\frac{\alpha}{2}}\,ds < \infty$, that is, iff

$\alpha \in (0, 2)$. To see this,

$$\begin{aligned}
L^\alpha &\subset LSub(H, p) \\
&\implies \int_0^1 \xi^{\frac{p}{2}} \rho(d\xi) < \infty \\
\implies \int_0^1 \xi^{\frac{p}{2}} \frac{1}{\beta\Gamma(1-\beta)\xi^{1+\beta}} \xi_{(0,\infty)}(\xi)\, d\xi &< \infty \\
\implies \frac{1}{\beta\Gamma(1-\beta)\xi^{1+\beta}} \int_0^1 \xi^{\frac{p}{2}} \xi_{(0,\infty)}(\xi)\, d\xi &< \infty
\end{aligned}$$

Now, with $p = 1$, $\beta = \frac{\alpha}{2}$

$$\begin{aligned}
\implies \frac{1}{\beta\Gamma(1-\beta)\xi^{1+\beta}} \int_0^1 \xi^{\frac{p}{2}} \xi_{(0,\infty)}(\xi)\, d\xi &< \infty \\
\implies \int_0^1 \frac{d\xi}{\xi^{-\frac{1}{2}+1+\frac{\alpha}{2}}} &< \infty
\end{aligned}$$

iff

$$-\frac{1}{2} + 1 + \frac{\alpha}{2} < 1 \implies \alpha < 1.$$

□

# CHAPTER 3

# Stochastic Navier-Stokes equations with cylindrical stable noise on 2D rotating spheres: weak solution, strong solution and invariant measure

## Summary

Our goal in this chapter is to generalise the analysis in Gaussian case [14] to the case where added noise is given by a stable Lévy noise. This chapter is concerned with the following stochastic Navier-Stokes equations (SNSE) on a 2D rotating sphere:

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \nabla p = f + \eta(x,t), \quad \text{div } u = 0, \quad u(0) = u_0 \tag{3.1}$$

where $\mathbf{L}$ is the stress tensor, $\omega$ is the Coriolis acceleration, $f$ is the external force and $\eta$ is the noise process that can be informally described as the derivative of an $H$-valued Lévy process. Rigorous definitions of all relevant quantities in this equation will be given in section 2 and 3. Our aim is to investigate the following three fundamental questions

- Does there exist weak solutions to (3.1) globally in time? Unique?
- Does there exist strong solutions to (3.1) globally in time? Unique?
- Does there exist stationary solutions to (3.1) time?

The new features in this chapter are the following. First, we prove that given $\mathbb{L}^4$-valued noise, $V'$-valued forcing $f$ and small $H$-valued initial data, there exists an uniqueness global weak (variational) solution which depends continuously on initial data. Moreover, with increased regularity of forcing and initial data, we prove an unique strong (PDE) solution for the abstract stochastic Navier-Stokes equations on the 2D unit sphere perturbed by stable Lévy noise. The existence time interval depends on the regularity of force and the assumption of the noise. Finally, deduce the existence of invariant measure for the SNSE and establish measure support.
The chapter is organised as follows. In section 3.1, we review the fundamental mathematical theory for the deterministic Navier-Stokes equations (NSE) on the sphere. In subsection 3.1.1, we recall some basic facts from spherical calculus and differential geometry. In subsection 3.1.2 we outline the necessary function spaces on the sphere in the theory of NSE on the

sphere. In subsection 3.1.3, we recall the weak formulation of these equations. In section 3.2, we define the SNSE on the spheres. We start with some analytic facts; we introduce the driving noise process, which is a stable Lévy noise via subordination. The SNSE is then decomposed into an Ornstein Uhlenbeck (OU) process (associated with the linear part of the SNSE) and a shift-invariant subset of full measure is identified that satisfies the Marcinkiewicz strong law of large number (see appendix). In section 3.3, we prove there exist global weak solution using the usual Galerkin approximation based on vector spherical harmonic series expansion. (see the proof of Theorem 3.2.5) Moreover, uniqueness is proven using the classical argument in the spirit of Lion and Prodi [74]. Furthermore, the solution is shown to depend continuously on initial data. (see the proof of Theorem 3.2.6) In section 3.4, we prove strong classical solution (see the proof of Theorem 3.3.7) for smooth initial data, sufficient regular noise following the classical lines in the proof of Theorem 3.1 [19]. In the final section, we prove the existence of an invariant measure.

## 3.1 Navier-Stokes equations on a rotating 2D unit sphere

The sphere is the simplest example of a compact Riemannian manifold without boundary hence one may employ the well-developed tools from Riemannian geometry to study objects on such manifold. Nevertheless, all objects of interests in this thesis are defined explicitly under the spherical coordinate. The presentation here follows closely from Goldys et al. [14] and reference therein.

### 3.1.1 Preliminaries

Let $\mathbb{S}^2$ be the 2D unit sphere in $\mathbb{R}^3$, that is $\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| = 1\}$. An arbitrary point $x$ on $\mathbb{S}^2$ can be parametrized by the spherical coordinates

$$x = \hat{x}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.$$

The corresponding angle $\theta$ and $\phi$ will be denoted by $\theta(x)$ and $\phi(x)$, or simply by $\theta$ and $\phi$.
Let $\mathrm{e}_\theta = \mathrm{e}_\theta(\theta, \phi)$ and $\mathrm{e}_\phi = \mathrm{e}_\phi(\theta, \phi)$ be the standard unit tangent vectors of $\mathbb{S}^2$ at point $\hat{x}(\theta, \phi) \in \mathbb{S}^2$ in the spherical coordinate, that is,

$$\mathrm{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad \mathrm{e}_\phi = (-\sin\phi, \cos\phi, 0).$$

Remark that

$$\mathrm{e}_\theta = \frac{\partial \hat{x}(\theta, \phi)}{\partial \theta}, \quad \mathrm{e}_\phi = \frac{1}{\sin\theta}\frac{\partial \hat{x}(\theta, \phi)}{\partial \phi},$$

where the second identity holds whenever $\sin\theta \neq 0$.
Our first aim is to give a meaning to all the terms in the deterministic Navier-Stokes equation

for the velocity field $u(\hat{x}, t) = (u_\theta(\hat{x}, t), u_\phi(\hat{x}, t))$ of a geophysical fluid flow on a 2D rotating unit sphere $\mathbb{S}^2$ under the external force $f = (f_\theta, f_\phi) = f_\theta e_\theta + f_\phi e_\phi$. Motion of the fluid is governed by the equation

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \frac{1}{\rho}\nabla p = f, \quad \nabla \cdot u = 0, \; u(x,0) = u_0. \tag{3.2}$$

Here $\nu$ and $\rho$ are two positive constants denote the viscosity and the density of the fluid, the normal vector field

$$\omega = 2\Omega \cos(\theta(x)) x,$$

where $x = \hat{x}(\theta(x), \phi(x))$ , $\Omega$ is the angular velocity of the earth and $\theta$ is the parameter represent the colatitude. Note that $\theta(x) = \cos^{-1}(x_3)$. In what follows we will identify $\omega$ with the corresponding scalar function $\omega$ defined by $\omega(x) = 2\Omega\cos(\theta(x))$.
We will introduce now other terms that appear in the equation.
The surface gradient for a scalar function $f$ on $\mathbb{S}^2$ is given by

$$\nabla f = \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{\sin\theta}\frac{\partial f}{\partial \phi} e_\phi, \quad 0 \le \theta \le \pi, \; 0 \le \phi \le 2\pi.$$

Unless specified otherwise, by a vector field on $\mathbb{S}^2$ we mean a tangential vector field, that is, a section of the tangent vector bundle of $\mathbb{S}^2$.
On the other hand, for a vector field $u = (u_\theta, u_\phi)$ on $\mathbb{S}^2$, that is $u = u_\theta e_\theta + u_\phi e_\phi$, one puts

$$\nabla \cdot u = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + \frac{\partial}{\partial\phi} u_\phi\right). \tag{3.3}$$

Given two vector fields $u$ and $v$ on $\mathbb{S}^2$, there exist vector fields $\tilde{u}$ and $\tilde{v}$ defined in some neighbourhood of the surface $\mathbb{S}^2$ and such that their restriction to $\mathbb{S}^2$ are equal to $u$ and $v$. More precisely, see Definition 3.31 in [44],

$$\tilde{u}|_{\mathbb{S}^2} = u : \mathbb{S}^2 \to T\mathbb{S}^2, \quad \text{and} \quad \tilde{v}|_{\mathbb{S}^2} = v : \mathbb{S}^2 \to T\mathbb{S}^2\,.$$

For $x \in \mathbb{R}^3$, we define the orthogonal projection $\pi_x : \mathbb{R}^3 \to T_x\mathbb{S}^2$ of $x$ onto $T_x\mathbb{S}^2$, that is

$$\pi_x : \mathbb{R}^3 \ni y \mapsto y - (x \cdot y)x = -x \times (x \times y) \in T_x\mathbb{S}^2. \tag{3.4}$$

**Lemma 3.1.1.** *Suppose $\tilde{u}$ and $\tilde{v}$ are $\mathbb{R}^3$-valued vector fields on $\mathbb{S}^2$, and $u$, $v$ are tangent vector field on $\mathbb{S}^2$, defined by $u(x) = \pi_x(\tilde{u}(x))$ and $v(x) = \pi_x(\tilde{v}(x))$ , $x \in \mathbb{S}^2$. Then the following identity holds*

$$\pi_x(\tilde{u}(x) \times \tilde{v}(x)) = u(x) \times ((x \cdot v(x))x) + ((x \cdot u(x))x \times v(x), \quad x \in \mathbb{S}^2. \tag{3.5}$$

**Proof.** The proof is in Goldys et al. [20], nevertheless, we include a summary of the proof here for readers' curiosity. Let us fix $x \in \mathbb{S}^2$. Then one may decompose vector $\tilde{u}$ and $\tilde{v}$ into tangential and normal components as follows

$$\tilde{u} = u + u^{\perp} \quad \text{with} \quad u \in T_x\mathbb{S}^2, \quad u^{\perp} = (u \cdot x)x,$$

$$\tilde{v} = v + v^{\perp} \quad \text{with} \quad v \in T_x\mathbb{S}^2, \quad v^{\perp} = (v \cdot x)x.$$

Since $u \times v$ is normal to $T_x\mathbb{S}^2$, $\pi_x(u \times v) = 0$. Likewise, $u^{\perp} \times v^{\perp} = 0$ since the cross product of two parallel vectors yields the 0 vector. Hence, it follows that

$$\pi_x(\tilde{u} \times \tilde{v}) = \pi_x(u \times v + u \times v^{\perp} + u^{\perp} \times v) = u \times v^{\perp} + u^{\perp} \times v \tag{3.6}$$

□

We will denote by $\tilde{\nabla}$ the usual gradient in $\mathbb{R}^3$ and then we have

$$(\nabla f)(x) = \pi_x(\tilde{\nabla}\tilde{f}(x)). \tag{3.7}$$

The operator **curl** is defined by the formula

$$(\mathbf{curl}\, u)(x) = (I - \pi_x)((\tilde{\nabla} \times \tilde{u})(x)) = (x \cdot (\tilde{\nabla} \times \tilde{u})(x))x. \tag{3.8}$$

Let $u$ be a tangent vector field on $\mathbb{S}^2$. Applying formula (3.6) to the vector fields $\tilde{u}$ and $\tilde{v} = \tilde{\nabla} \times \tilde{u}$, one gets

$$\begin{aligned} \pi_x(\tilde{u} \times (\tilde{\nabla} \times \tilde{u})) &= \tilde{u} \times (\tilde{\nabla} \times (u^{\perp} + u) \\ &= u \times ((\nabla \times u)^{\perp}) + u^{\perp} \times (\nabla \times u) \\ &= u \times ((x \cdot (\tilde{\nabla} \times \tilde{u}))x) \\ &= (x \cdot (\tilde{\nabla} \times \tilde{u}))(u \times x), \quad x \in \mathbb{S}^2. \end{aligned} \tag{3.9}$$

So, we can now define the curl of the vector field $u$ on $\mathbb{S}^2$, namely,

$$\text{curl } u := \hat{x} \cdot (\tilde{\nabla} \times \tilde{u})|_{\mathbb{S}^2} \tag{3.10}$$

equations (3.6) and (3.10) yield

$$\pi_x[\tilde{u} \times (\tilde{\nabla} \times \tilde{u})](x) = [u(x) \times x]\, \text{curl} u(x), \quad x \in \mathbb{S}^2$$

Therefore, we have the following

**Definition 3.1.2.** Let $u$ be a tangent vector field on $\mathbb{S}^2$, and let the vector field $\psi$ be normal to $\mathbb{S}^2$. We set

$$\text{curl}\, u = (\hat{x} \cdot (\tilde{\nabla} \times \tilde{u}))|_{\mathbb{S}^2}, \quad \text{Curl}\, \psi = (\tilde{\nabla} \times \tilde{u})|_{\mathbb{S}^2} \tag{3.11}$$

The first equation above indicates a projection of $\nabla \times \tilde{u}$ onto the normal direction, while the 2nd equation means a restriction of $\nabla \times \psi$ to the tangent field on $\mathbb{S}^2$. The definitions presented above do not depend on the extensions $\tilde{u}$ and $\tilde{\psi}$. A vector field $\psi$ normal to $\mathbb{S}^2$ will often be identified with a scalar function on $\mathbb{S}^2$ when it is convenient to do so. The following describe the relationships among Curl of a scalar function $\psi$, Curl of a normal vector field $\mathbf{w} = w\hat{x}$, and curl of a vector field $v$ on $\mathbb{S}^2$ and the surface div and $\Delta$ operators are given as

$$\mathrm{Curl}\,\psi = -\hat{x} \times \nabla\psi, \quad \mathrm{Curl}\,\mathbf{w} = -\hat{x} \times \nabla w, \quad \mathrm{curl}\ v = -\mathrm{div}(\hat{x} \times v). \tag{3.12}$$

Let

$$(\nabla_v u)(x) = \pi_x \left( \sum_{i=1}^{3} \tilde{v}_i(x) \partial_i \tilde{u}(x) \right) = \pi_x \left( (\tilde{v}(x) \cdot \tilde{\nabla}) \tilde{u}(x) \right), \quad x \in \mathbb{S}^2. \tag{3.13}$$

Invoking (3.5) and the formula

$$(\tilde{u} \cdot \tilde{\nabla})\tilde{u} = \tilde{\nabla} \frac{|\tilde{u}^2|}{2} - \tilde{u} \times (\tilde{\nabla} \times \tilde{u}),$$

we find that the covariant derivative $\nabla_u u$ takes the form

$$\nabla_u u = \nabla \frac{|u^2|}{2} - \pi_x(\tilde{u} \times (\tilde{\nabla} \times \tilde{u})).$$

In particular, using (3.5) we obtain

$$\nabla_u u = \nabla \frac{|u|^2}{2} - u \times \mathrm{curl} u.$$

The surface diffusion operator acting on vector fields on $\mathbb{S}^2$ is denoted by $\boldsymbol{\Delta}$ (known as the Laplace de Rham operator) and is defined as

$$\boldsymbol{\Delta} v = \nabla \mathrm{div}\ v - \mathrm{Curl}\ \mathrm{curl}\ v. \tag{3.14}$$

Using (3.12), one can derive the following relations connecting the above operators:

$$\mathrm{div}\ \mathrm{Curl}\ v = 0, \quad \mathrm{curl}\ \mathrm{Curl}\ \mathrm{v} = -\hat{x}\Delta v, \quad \boldsymbol{\Delta}\mathrm{Curl}\ v = \mathrm{Curl}\Delta v. \tag{3.15}$$

Next, we recall the definition of the Ricci tensor Ric of the 2D sphere $\mathbb{S}^2$. Since

$$\mathrm{Ric} = \begin{pmatrix} E & F \\ F & C \end{pmatrix}$$

where the coefficients $E, F, G$ of the first fundamental form are given by

$$E = x_\theta \cdot x_\theta = 1$$

$$F = x_\theta \cdot x_\phi = x_\phi \cdot x_\theta = 0$$

$$C = x_\phi \cdot x_\phi = \sin^2\theta$$

we find that

$$\mathrm{Ric} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{3.16}$$

Finally we define the stress tensor $\mathbf{L}$: it is given by

$$\mathbf{L} = \mathbf{\Delta} + 2\mathrm{Ric}$$

where $\mathbf{\Delta}$ is the Laplace-de Rham operator.

### 3.1.2 Function spaces on the sphere

In what follows we denote by $dS$ the surface measure on $\mathbb{S}^2$. In the spherical coordinate one has locally, $dS = \sin\theta\, d\theta d\phi$. For $p \in [1,\infty)$ we denote by $L^p = L^p(\mathbb{S}^2,\mathbb{R})$ of $p$ integrable scalar function on $\mathbb{S}^2$ endowed with the norm

$$|v|_{L^p} = \left(\int_{\mathbb{S}^2} |v(x)|^p dS(x)\right)^{1/p}.$$

For $p = 2$ the corresponding inner product is denoted by

$$(v_1, v_2) = (v_1, v_2)_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} v_1 v_2\, dS$$

On the other hand, we denote $\mathbb{L}^p = \mathbb{L}^p(\mathbb{S}^2)$ the space $L^p(\mathbb{S}^2, T\mathbb{S}^2)$ of vector fields $v : \mathbb{S}^2 \to T\mathbb{S}^2$ endowed with the norm

$$|v|_{L^p} = \left(\int_{\mathbb{S}^2} |v(x)|^p dS(x)\right)^{1/p},$$

where, for $x \in \mathbb{S}^2$, $|v(x)|$ denotes the length of $v(x)$ in the tangent space $T_x\mathbb{S}^2$. For $p = 2$ the corresponding inner product is denoted by

$$(v_1, v_2) = (v_1, v_2)_{\mathbb{L}^2} = \int_{\mathbb{S}^2} v_1 \cdot v_2\, dS.$$

Throughout this thesis, the induced norm on $\mathbb{L}^2(\mathbb{S}^2)$ is denoted by $|\cdot|$. For other inner product spaces, say $V$ with inner product $(\cdot,\cdot)_V$, the associated norm is denoted by $|\cdot|_V$.
The following identities hold for appropriate real valued scalar functions and vector fields on $\mathbb{S}^2$, see (2.4)-(2.6) [64]:

$$(\nabla\psi, v) = -(\psi, \mathrm{div} v), \tag{3.17}$$

$$(\mathrm{Curl}\ \psi, v) = (\psi, \mathrm{curl}\ v), \tag{3.18}$$

$$(\mathrm{Curlcurl}\ w, z) = (\mathrm{curl}\ w, \mathrm{curl}\ z). \tag{3.19}$$

In (3.18), the $\mathbb{L}^2(\mathbb{S}^2)$ inner product is used on the left hand side while the $L^2(\mathbb{S}^2)$ is used on the right hand side. Throughout this thesis, we identify a normal vector field $\mathbf{w}$ with a scalar field

$w$ and by $\mathbf{w} = \hat{x}w$ and hence we put

$$(\psi, \mathbf{w}) := (\psi, w)_{L^2(\mathbb{S}^2)}, \quad \text{if } \mathbf{w} = \hat{x}w, \quad \psi, w \in L^2(\mathbb{S}^2). \tag{3.20}$$

Let us now introduce the Sobolev spaces $H^1(\mathbb{S}^2)$ and $\mathbb{H}^1(\mathbb{S}^2)$ of scalar functions and vector fields on $\mathbb{S}^2$. Let $\psi$ be a scalar function and let $u$ be a vector field on $\mathbb{S}^2$, respectively. For $s \geq 0$ we define

$$|\psi|^2_{H^1(\mathbb{S}^2)} = |\psi|^2_{L^2(\mathbb{S}^2)} + |\nabla\psi|^2_{L^2(\mathbb{S}^2)}, \tag{3.21}$$

and

$$|u|^2_{\mathbb{H}^1(\mathbb{S}^2)} = |u|^2 + |\nabla \cdot u|^2 + |\mathrm{Curl}\, u|^2 . \tag{3.22}$$

One has the following Poincaré inequality

$$\lambda_1|u| \leq |\mathrm{div} u| + |\mathrm{Curl} u|, \quad u \in \mathbb{H}^1(\mathbb{S}^2), \tag{3.23}$$

where $\lambda_1 > 0$ is the first positive eigenvalue of the Laplace-Hodge operator, see below. By the Hodge decomposition theorem in Riemannian geometry [31], the space of $C^\infty$ smooth vector field on $\mathbb{S}^2$ can be decomposed into three components:

$$C^\infty(T\mathbb{S}^2) = \mathcal{G} \oplus \mathcal{V} \oplus \mathcal{V},$$

where

$$\mathcal{G} = \{\nabla\psi \in C^\infty(\mathbb{S}^2), \quad \mathcal{V} = \{\mathrm{Curl}\psi \in C^\infty(\mathbb{S}^2),$$

and $\mathcal{H}$ is the finite-dimensional space of harmonic vector fields. Since the sphere is simply connected, that is, the map $\mathbb{S}^2 \to \mathbb{S}^2$ is a diffeomorphism and so $\mathcal{H} = \{0\}$. The condition of orthogonality to $\mathcal{H}$ is dropped out. We introduce the following spaces

$$\begin{aligned} H &:= \{u \in \mathbb{L}^2(\mathbb{S}^2) : \nabla \cdot u = 0\}, \\ V &:= H \cap \mathbb{H}^1(\mathbb{S}^2). \end{aligned}$$

In other words, $H$ is the closure of the

$$\{u \in C^\infty(T\mathbb{S}^2) : \nabla \cdot u = 0\}$$

in the $\mathbb{L}^2$ norm $|u| = (u, u)^{1/2}$, where $u = (u_\theta, u_\phi)$ and

$$(u, v) = \int_{\mathbb{S}^2} u_\theta(x) v_\theta(x) dx, \tag{3.24}$$

and the space $V$ is the closure of

$$\{u \in C^\infty(T\mathbb{S}^2) : \nabla \cdot u = 0\}$$

in the norm of $\mathbb{H}^1\left(\mathbb{S}^2\right)$. Since $V$ is densely and continuously embedded into $H$ and $H$ can be identified with its dual $H'$, one has the following Gelfand triple:

$$V \subset H \cong H' \subset V'. \tag{3.25}$$

### 3.1.3 Stokes operator

We will recall first that the Laplace-Beltrami operator on $\mathbb{S}^2$ can be defined in terms of spherical harmonics $Y_{l,m}$ as follows. For $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$, we define

$$Y_{l,m}(\theta,\varphi) = \left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\varphi}, \quad m = -l,\cdots,l, \tag{3.26}$$

with $P_l^m$ being the associated Legendre polynomials. The family $\{Y_{l,m} : l = 0,1,\dots,\ m = -l,\dots,l\}$ form an orthonormal basis in $L^2\left(\mathbb{S}^2\right)$ and then we can define the Laplace-Beltrami operator putting

$$\Delta Y_{l,m} = -l(l+1)Y_{l,m}$$

, and then extending by linearity to all functions $f : L^2\left(\mathbb{S}^2\right)$ such that

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} l^2(l+1)^2 \left(f, Y_{l,m}\right)^2_{L^2\left(\mathbb{S}^2\right)} < \infty\,.$$

We consider the following linear Stokes problem, that is given $f \in V'$, find $v \in V$ such that

$$\nu \mathrm{Curlcurl} u - 2\nu \mathrm{Ric}(u) + \nabla p = f, \quad \nabla \cdot u = 0. \tag{3.27}$$

By taking the inner product of the first equation above with a test field $v \in V$ and then use (3.19), the pressure term drops and we obtain

$$\nu(\mathrm{curl} u, \mathrm{curl} v) - 2\nu(\mathrm{Ric}\, u, v) = (f, \mathbf{v}) \quad \forall\, v \in V.$$

Next, define a bilinear form $a : V \times V \to \mathbb{R}$ by

$$a(u,v) := (\mathrm{curl}\, u, \mathrm{curl}\, v) - 2(\mathrm{Ric}\, u, v), \quad u,\ v \in V. \tag{3.28}$$

In view of (3.22) and the formula (3.16) for the Ricci tensor on $\mathbb{S}^2$, the bilinear form $a$ satisfies

$$a(u,v) \leq |u|_{\mathbb{H}^1}|v|_{\mathbb{H}^1} \tag{3.29}$$

and so it is continuous on $V$. So, by the Riesz representation theorem, there exists a unique operator $\mathcal{A} : V \to V'$ where $V'$ is the dual of $V$, such that $a(u,v) = (\mathcal{A}u, v)$, for $u, v \in V$. Invoking the Poincaré inequality (3.23) we find that $a(u,u) \geq \alpha|u|_V^2$, for a certain $\alpha > 0$, which implies that $a$ is coercive in $V$. Hence by the Lax-Milgram theorem the operator $\mathcal{A} : V \to V'$

is an isomorphism. Let $A$ be a restriction of $\mathcal{A}$ to $H$:

$$\begin{cases} D(A) & := \{u \in V : \mathcal{A}u \in H\}, \\ Au & := \mathcal{A}u, \quad u \in D(A). \end{cases} \tag{3.30}$$

It is well known (see for instance [102], Theorem 2.2.3 ) that $A$ is positive definite, self-adjoint in $H$ and $D(A^{1/2}) = V$ with equivalent norms. Furthermore, for some positive constants $c_1, c_2$ we have

$$c_1|u|_{D(A)} \leq |Au| \leq c_2|u|_{D(A)}\,,$$

$$\langle Au, u\rangle = ((u,u)) = |u|_V = |\nabla u|^2 = |Du|^2, \quad u \in D(A). \tag{3.31}$$

The spectrum of A consists of an infinite sequence of eigenvalues $\lambda_l$. Using the stream function $\psi_l$ for which $\mathbf{Z}_{lm} = \mathrm{Curl}\psi_{l,m}$ and identities (3.15), one can show that each $\lambda_l$ are in fact the eigenvalues of the Laplace-Beltrami operator $\Delta$, that is $\lambda_l = l(l+1)$, and there exists an orthonormal basis $(\mathbf{Z}_{l,m})_{l\geq 1}$ of $H$ consisting of eigenvector of $A$, where

$$\mathbf{Z}_{l,m} = \lambda_l^{-1/2}\mathrm{Curl}Y_{l,m}, \quad l = 1,\ldots, m = -l,\ldots,l, \tag{3.32}$$

Therefore, for any $v \in H$, one has,

$$v = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \hat{v}_{l,m}\mathbf{Z}_{l,m}, \quad \widehat{v}_{l,m} = \int_{\mathbb{S}^2} v\cdot \mathbf{Z_{l,m}}dS = (v, \mathbf{Z_{l,m}}). \tag{3.33}$$

An equivalent definition of the operator $A$ can be given using the so-called Leray-Helmhotz projection $P$ that is defined as an orthogonal projection from $\mathbb{L}^2(\mathbb{S}^2)$ onto $H$, called Leray-Helmhotz projection. Let $\mathbb{H}^2(\mathbb{S}^2)$ denote the domain of the Laplace-Hodge operator in $H$ endowed with the graph norm. It can be shown in [60] that $D(A) = \mathbb{H}^2(\mathbb{S}^2) \cap V$ and $A = -P(\mathbf{\Delta} + 2\mathrm{Ric})$. Therefore, we obtain an equivalent definition of the Stokes operator on the sphere.

**Definition 3.1.3.** The Stokes operator $A$ on the sphere is defined as

$$A : D(A) \subset H \to H, \quad A = -P(\mathbf{\Delta} + 2\mathrm{Ric}), \quad D(A) = \mathbb{H}^2(\mathbb{S}^2) \cap V, \tag{3.34}$$

where $\mathbf{\Delta}$ is the Laplace-De Rham operator.

It can be shown that $V = D(A^{1/2})$ when endowed with the norm $|x|_V = |A^{1/2}x|$ and the inner product $((x,y)) = \langle Ax, y\rangle$. After identification of $H$ with its dual space we have $V \subset H \subset V'$ with continuous dense injection. The dual pairing between $V$ and $V'$ is denoted by $(\cdot,\cdot)_{V\times V'}$. Moreover, there exist positive constants $c_1$, $c_2$ such that

$$c_1|u|_V^2 \leq (Au, u) \leq c_2|u|_V^2, \quad u \in D(A).$$

Let us now introduce the Sobolev spaces $H^s(\mathbb{S}^2)$ and $\mathbb{H}^2(\mathbb{S}^2)$ of scalar functions and vector fields on $\mathbb{S}^2$. Let $\psi$ be a scalar function and let $u$ be a vector field on $\mathbb{S}^2$, respectively. For $s \geq 0$ we define

$$|\psi|^2_{H^s(\mathbb{S}^2)} = |\psi|^2_{L^2(\mathbb{S}^2)} + |(-\Delta)^{s/2}\psi|^2_{L^2(\mathbb{S}^2)}, \tag{3.35}$$

and

$$|u|^2_{\mathbb{H}^s(\mathbb{S}^2)} = |u|^2 + |(-\boldsymbol{\Delta})^{s/2}u|^2, \tag{3.36}$$

where $\Delta$ is the Laplace-Beltrami operator and $\boldsymbol{\Delta}$ is the Laplace-de Rham operator on the sphere. Note that, for $k = 0, 1, 2, \cdots$ and $\theta \in (0, 1)$ the space $H^{k+\theta}(\mathbb{S}^2)$ can be defined as the interpolation space between $H^k(\mathbb{S}^2)$ and $H^{k+1}(\mathbb{S}^2)$. One can apply the same procedure for $H^{k+\theta}(\mathbb{S}^2)$, [20]. The fractional power $A^{s/2}$ of the Stokes operator $A$ in $H$ for any $s \geq 0$ is given by

$$D(A^{s/2}) = \left\{ v \in H : v = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \hat{v}_{l,m}\mathbf{Z}_{l,m},\ \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \lambda_l^s|\hat{v}_{l,m}|^2 < \infty \right\},$$

$$A^{s/2}v := \sum_{m=1}^{\infty}\sum_{m=-l}^{l} \lambda_l^{s/2}\hat{v}_{l,m}\mathbf{Z}_{l,m} \in H.$$

The Coriolis operator $\mathbf{C}_1 : \mathbb{L}^2(\mathbb{S}^2) \to \mathbb{L}^2(\mathbb{S}^2)$ is defined by the formula[1]

$$(\mathbf{C}_1 v)(x) = 2\Omega(x \times v(x))\cos\theta, \quad x \in \mathbb{S}^2. \tag{3.37}$$

It is clear from the above definition that $\mathbf{C}_1$ is a bounded linear operator defined on $\mathbb{L}^2(\mathbb{S}^2)$. In the sequel we will need the operator $\mathbf{C} = P\mathbf{C}_1$ which is well defined and bounded in $H$. Furthermore, for $u \in H$,

$$(\mathbf{C}u, u) = (\mathbf{C}_1 u, Pu) = \int_{\mathbb{S}^2} 2\Omega\cos\theta((x \times u) \cdot u(x))dS(x) = 0. \tag{3.38}$$

In addition,

**Lemma 3.1.4.** *For any smooth function $u$ and $s \geq 0$*

$$(\mathbf{C}u, A^s u) = 0. \tag{3.39}$$

**Proof.** The case $s = 0$ is obvious as in the line above, due to the fact $(\omega \times u) \cdot u = 0$. For $s > 0$ we refer readers to Lemma 5 in [99]. □

[1]The angular velocity vector of earth is denoted as $\Omega$ in consistant to geophysical fluid dynamics Literature. It shall not be confused with the notation for probility space $\Omega$ used in this thesis.

Let $X = H \cap \mathbb{L}^4\left(\mathbb{S}^2\right)$ be endowed with the norm

$$|v|_X = |v|_H + |v|_{\mathbb{L}^4\left(\mathbb{S}^2\right)}.$$

Then $X$ is a Banach space. It is known that the Stokes operator $A$ generates an analytic $C_0$-semigroup $\{e^{-tA}\}_{t\geq 0}$ in $X$ (see Theorem A.1 in [14]). Since the Coriolis operator $\mathbf{C}$ is bounded on $X$ we can define in $X$ an operator

$$\hat{A} = \nu A + \mathbf{C}, \quad D(\hat{A}) = D(A),$$

with $\nu > 0$.

**Lemma 3.1.5.** *Suppose that $V \subset H \cong H' \subset V'$ is a Gelfand triple of Hilbert spaces. If a function $u$ being $L^2(0,T;V)$ and $\partial_t u$ belongs to $L^2(0,T;V')$ in weak sense, then $u$ is a.e. equal to a continuous function from $[0,T]$ to $H$, the real function $|u|^2$ is absolutely continuous and, in the weak sense one has*

$$\partial_t |u(t)|^2 = 2\langle \partial_t u(t), u(t)\rangle \tag{3.40}$$

**Proposition 3.1.6.** *The operator $\hat{A}$ with the domain $D(\hat{A}) = D(A)$ generates a strongly continuous and analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in $X$. In particular, there exist $M \geq 1$ and $\mu > 0$ such that*

$$|e^{-t\hat{A}}|_{\mathcal{L}(X,X)} \leq Me^{-\mu t}, \quad t \geq 0\,, \tag{3.41}$$

*and for any $\delta > 0$ there exists $M_\delta \geq 1$ such that*

$$|\hat{A}^\delta e^{-t\hat{A}}|_{\mathcal{L}(X,X)} \leq M_\delta t^{-\delta} e^{-\mu t}, \quad t > 0. \tag{3.42}$$

**Proof.** The proof can be found in [14] (the proof of Proposition 5.3). Nevertheless, we include the proof here for readers' convenience. Since $A$ is the infinitesimal generator of the analytic $C_0$-semigroup in $X$, $C$ is the bounded linear operator on $X$, then by Theorem A.1 [14] and Corollary 2.2 on p.81 of [86] we infer that operator $\hat{A}$ is a generator of another analytic $C_0$ semigroup on $X$. Suppose $u(t) = e^{-t(\nu A + C)}u_0$ for some $u_0 \in V \subset X$. Let us first show that

$$\int_0^\infty |u(t)|^2 dt = \int_0^\infty |e^{-t(\nu A+C)}u_0|^2_{\mathbb{L}^2(\mathbb{S}^2)}\, dt < \infty. \tag{3.43}$$

By Lemma 3.1.5 and identity (3.38), we have

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 = (u'(t), u(t)) = -(\nu A u, u) - (\mathbf{C}u, u) = -\nu|u|^2_V.$$

Hence

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|u|_V^2 = 0. \tag{3.44}$$

Since $|u|_V^2 = |A^{1/2}u|^2 \geq \lambda_1|u|^2$ by Lemma 3.23, we obtain

$$\frac{d}{dt}|u(t)|^2 \leq -2\lambda_1\nu|u(t)|^2.$$

Using the Gronwall inequality, we obtain $|u(t)|^2 \leq e^{-2\lambda_1\nu t}|u_0|^2$ for $u_0 \in \mathbb{H}^2(\mathbb{S}^2)$, hence for all $u_0 \in X$, the claim (3.43) follows.

Now integrate (3.44) from 0 to $T$ we have

$$|u(T)|^2 + 2\nu\int_0^T |u(t)|_V^2 dt = |u(0)|^2.$$

It is clear that

$$\int_0^\infty |u(t)|_V^2 < \infty \tag{3.45}$$

as $T \to \infty$ for all $u_0 \in H$. Using the interpolation inequality in p.12 [66], the Cauchy-Schwartz inequality and the previous three inequalities we obtain that

$$\int_0^\infty |u(t)|_{\mathbb{L}^4}^2 dt \leq \left(\int_0^\infty |u(t)|_{\mathbb{L}^2}^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty |u(t)|_V^2 dt\right)^{\frac{1}{2}} < \infty.$$

Invoking Theorem 4.1 on p.116 [86] with $X = H \cap \mathbb{L}^4(\mathbb{S}^2)$ we conclude that $|e^{-t\hat{A}}|_{\mathcal{L}(X,X)} \leq Me^{-\mu t}$ for some constants $M \geq 1$ and $\mu > 0$. Finally, using Theorem 6.13 on p.74 of [86] with $X = H \cap \mathbb{L}^4(\mathbb{S}^2)$, the conclusion in (3.42) follows. □

Now consider the trilinear form $b$ on $V \times V \times V$, defined as

$$b(v, w, z) = (\nabla_v w, z) = \int_{\mathbb{S}^2} \nabla_v w \cdot z dS = \pi_x \sum_{i,j=1}^{3} v_j D_i w_j z_j dx, \quad v, w, z \in V. \tag{3.46}$$

Via the identity [14],

$$2\nabla_w v = -\text{curl}(w \times v) + \nabla(w \cdot v) - v\,\text{div}\, w + w\,\text{div}\, v - v \times \text{curl}\, w - w \times \text{curl}\, v,$$

and equation (3.14), one can write the divergence free fields $v, w, z$, the trilinear form can be written as

$$b(v, w, z) = \frac{1}{2}\int_{\mathbb{S}^2}[-v \times w \cdot \text{curl}\, z + \text{curl}\, v \times w \cdot z - v \times \text{curl}\, w \cdot z]dS. \tag{3.47}$$

Moreover,

$$b(v, w, w) = 0, \quad b(v, z, w) = -b(v, w, z), \quad v \in V, w, z \in \mathbb{H}^1(\mathbb{S}^2), \tag{3.48}$$

and such that

$$|B(u,v),w| = |b(u,v,w)| \le c|u||w|(|\mathrm{curl}\ v|_{\mathbb{L}^\infty(\mathbb{S}^2)} + |v|_{\mathbb{L}^\infty(\mathbb{S}^2)}), \quad u \in H, v \in V, w \in H, \tag{3.49}$$

$$|B(u,v),w| = |b(u,v,w)| \le c|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2}|w|_V, \quad u,v,w \in V, \tag{3.50}$$

$$|B(u,v),w| = |b(u,v,w)| \le c|u|^{1/2}|u|_V^{1/2}|v|_V^{1/2}|Au|^{1/2}|w|,\ \forall\, u \in V, v \in D(A), w \in H,\ n=2, \tag{3.51}$$

$$|b(u,v,w)| \le c|u|_{\mathbb{L}^4(\mathbb{S}^2)}|v|_V|w|_{\mathbb{L}^4(\mathbb{S}^2)}, \quad v \in V, u,w \in \mathbb{H}^1(\mathbb{S}^2). \tag{3.52}$$

In view of (3.50),

$$\begin{aligned} \sup_{z\in V, |z|_V \neq 0} \frac{|(B(u,v),z)|}{|z|_V} = |B(u,v)|_{V'} &\le c|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2} \\ \Longrightarrow |B(u,u)|_{V'} &\le c|u||u|_V, \\ |B(u,u)|_H &\le c|u||u|_V. \end{aligned} \tag{3.53}$$

$$\begin{aligned} \sup_{z\in H, |z|_H \neq 0} \frac{|(B(u,v),z)|}{|z|_H} = |B(u,v)|_H &\le c|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2} \\ \Longrightarrow |B(u,u)|_H &\le c|u||u|_V. \end{aligned} \tag{3.54}$$

In view of (3.51),

$$\begin{aligned} &\sup_{z\in H, |z|_H \neq 0} \frac{|(B(u,v),z)|}{|z|_H} = |B(u,v)|_H \le c|u|^{1/2}|u|_V^{1/2}|u|^{1/2}|Au|^{1/2} \\ &\Longrightarrow |B(u,u)|_H \le c|u|^{1/2}|u|_V|Au|^{1/2} \le c|u|_V^{1/2}|u|_V|Au|^{1/2} \quad \forall\ u \in D(A). \end{aligned} \tag{3.55}$$

In view of (3.52), $b$ is a bounded trilinear map from $\mathbb{L}^4(\mathbb{S}^2) \times V \times \mathbb{L}^4(\mathbb{S}^2)$ to $\mathbb{R}$.

**Lemma 3.1.7.** *The trilinear map $b$ can be uniquely extended from $V \times V \times V$ to a bounded three-linear map*

$$b : (\mathbb{L}^4(\mathbb{S}^2) \cap H) \times \mathbb{L}^4(\mathbb{S}^2) \times V \to \mathbb{R}\,.$$

Finally, we recall the interpolation inequality (See [66], p.12),

$$|u|_{\mathbb{L}^4(\mathbb{S}^2)} \le C|u|_{\mathbb{L}^2(\mathbb{S}^2)}^{1/2}|u|_V^{1/2}\,. \tag{3.56}$$

Inequality (3.50) is deduced from the following Sobolev embedding

$$H^{1/2} = W^{1/2,2}(\mathbb{S}^2) \hookrightarrow \mathbb{L}^4(\mathbb{S}^2).$$

Then using (3.14), (3.17), (3.30) and (3.47), we arrive with the *weak solution* of the Navier-Stokes equations (3.3), which is a vector field $u \in L^2([0,T];V)$ with $u(0) = u_0$ that satisfies the weak form of (3.3):

$$(\partial_t u, v) + b(u,u,v) + \nu(\text{curl}u, \text{curl}v) - 2\nu(\text{Ric } u, v) + (\mathbf{C}u, v) = (f,v), \quad v \in V, \tag{3.57}$$

where the bilinear form $B : V \times V \to V'$ is defined by

$$(B(u,v),w) = b(u,v,w) = \sum_{i,j=1}^{3} \int_{\mathbb{S}^2} u_i \frac{\partial (v_k)_j}{\partial x_i} u_j dx, \quad w \in V. \tag{3.58}$$

With a slight abuse of notation, we denote $B(u) = B(u,u)$ and $B(u) = \pi(u, \nabla u)$.

## 3.2 Stochastic Navier-Stokes equations on the 2D unit sphere

By adding a Lévy white noise to (3.2), we obtain the main equation in this thesis.

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \nabla p = f + \eta(x,t), \tag{3.59}$$

$$\text{div } u = 0, u(x,0) = u_0, \; x \in \mathbb{S}^2. \tag{3.60}$$

We assume that, $u_0 \in H$, $f \in V'$ and $\eta(x,t)$ is the so-called Lévy white noise, that is a noise process which can be informally described as the derivative of an $H$-valued Lévy process, that is rigorously defined in Lemma 3.2.7. Applying the Leray-Helmholz projection we can interpret equation (3.59) as an abstract stochastic equation in $H$

$$du(t) + Au(t) + B(u(t),u(t)) + \mathbf{C}u = fdt + GdL(t), \quad u(0) = u_0, \tag{3.61}$$

where $L$ is an $H$-valued stable Lévy process and $G : H \to H$ is a bounded operator. In order to study this equation we need to consider first some properties of the stochastic convolution.

### 3.2.1 Stochastic convolution of $\beta$-stable noise

In this section we will study a linear version of equation (3.61)

$$dz(t) + Az(t) + \mathbf{C}z = GdL(t), \quad z(0) = 0\,. \tag{3.62}$$

Under appropriate assumptions formulated below its solution takes the form

$$z_t = \int_0^t e^{-\widehat{A}(t-s)} GdL(s), \tag{3.63}$$

where $\widehat{A} = A + \mathbf{C}$. Let $W$ be a cylindrical Wiener process on a Hilbert space $K$ continuously imbedded into $H$ and let $X$ be a $\beta/2$-stable subordinator. Then the process $L = W(X)$ is a symmetric cylindrical $\beta$-stable process in $H$. Assume that $G : H \to H$ is $\gamma$-radonifying. Then the process $GL$ is a well defined Lévy process taking values in $H$. Under these assumptions the process $z$ defined by (3.63) is a well defined $H$-valued process and moreover, it can be considered as a solution to the following integral equation

$$z(t) = -\int_0^t e^{-(t-s)A}\mathbf{C}z(s)\,ds + \int_0^t e^{-(t-s)A}G\,dL(s) \tag{3.64}$$

With some abuse of notation, we will denote now by $\lambda_l$ the eigenvalues of the Stokes operator $A$ taking into account their mulitplicities that is $\lambda_1 \le \lambda_2 \le \cdots$, and by $e_l$ the corresponding eigenvectors that form an orthonormal basis in $H$. We will impose a stronger condition on the operator $G$:

$$Ge_l = \sigma_l e_l, \quad l = 1, 2, \dots .$$

We will consider the process

$$z_t^0 = \int_0^t e^{-(t-s)A}GdL(s) = \sum_{l=1}^{\infty} z_l^0(t)e_l,$$

where

$$z_l^0(t) = \int_0^t e^{-\lambda_l(t-s)}\sigma_l dL_l(s). \tag{3.65}$$

**Lemma 3.2.1.** *Suppose that there exists some $\delta > 0$ such that $\sum_{l\ge1} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty$. Then for all $p \in (0, \beta)$,*

$$\mathbb{E}|A^\delta L(t)|^p \le C(\beta, p)\left(\sum_{l\ge1} |\sigma_l|^\beta \lambda_l^{\beta\delta}\right)^{\frac{p}{\beta}} t^{\frac{p}{\beta}} < \infty. \tag{3.66}$$

**Proof.** Let $L(t) = \sum_{l\ge1} L_t^l e_l$, $t \ge 0$ be the cylindrical $\beta$-stable process on $H$, where $e_l$ is the complete orthonormal system of eigenfunctions on $H$ and $L_1, L_2, \cdots, L_l$ are i.i.d. $\mathbb{R}$-valued, symmetric $\beta$-stable process on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now take a bounded sequence of real number $\sigma = (\sigma_l)_{l\in\mathbb{N}}$, let us define

$$G_\sigma : H \to H; \quad G_\sigma u := \sum_{l=1}^{\infty} \sigma_l \langle u, e_l\rangle e_l,$$

and $\sigma_l$ are chosen such that

$$G_\sigma L(t) = \sum_{l=1}^{\infty} \sigma_l \langle L_l(t), e_l\rangle e_l = \sum_{l=1}^{\infty} \sigma_l L_l(t) e_l.$$

To show (3.66), we follow the argument in the proof of Lemma 3.1 in [115] and Theorem 4.4 in [90]. Take a Rademacher sequence $\{r_k\}_{k\geq 1}$ in a new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, that is, $\{r_k\}_{k\geq 1}$ are i.i.d. with $\mathbb{P}\{r_k = 1\} = \mathbb{P}\{r_k = -1\} = \frac{1}{2}$. By the following Khintchine inequality: for any $p > 0$, there exists some $C(p) > 0$ such that for arbitrary real sequence $\{h_l\}_{l\geq 1}$,

$$\left(\sum_{l\geq 1} h_l^2\right)^{1/2} \leq C(p)\left(\mathbb{E}'|\sum_{l\geq 1} r_l h_l|^p\right)^{1/p}.$$

Via this inequality, we get

$$\begin{aligned}\mathbb{E}|A^\delta L(t)|^q &= \mathbb{E}\left(\sum_{l\geq 1}\lambda_l^{2\delta}|\sigma_l|^2|L_l(t)|^2\right)^{p/2}\\ &\leq C\mathbb{E}\mathbb{E}'\left|\sum_{l\geq 1} r_l\lambda_l^\delta|\sigma_l||L_l(t)|\right|^p\\ &= C\mathbb{E}'\mathbb{E}\left|\sum_{l\geq 1} r_l\lambda_l^\delta|\sigma_l||L_l(t)|\right|^p,\end{aligned}$$

where $C = C^p(p)$. For any $\lambda \in \mathbb{R}$, by the fact of $|r_k| = 1$ and formula (4.7) of [90],

$$\mathbb{E}\exp\left\{i\eta\sum_{l\geq 1} r_l\eta_l^\delta|\sigma_l|L_l(t)\right\} = \exp\left\{-|\eta|^\delta\sum_{l\geq 1}|\sigma_l|^\beta\lambda_l^{\beta\delta}t\right\}.$$

Now we know that any symmetric $\beta$-stable r.v. $X \sim \tilde{S}_\alpha(\sigma, 0, 0)$ satisfies

$$\mathbb{E}e^{i\eta X} = e^{-\sigma^\beta\eta^\beta}$$

for some $\beta \in (0, 2)$, $\eta \in \mathbb{R}$, then for any $p \in (0, \beta)$,

$$\mathbb{E}|X|^p = C(\beta, p)\sigma^p.$$

Since $\sum_{l\geq 1}|\sigma_l|^\beta\lambda_l^{\beta\delta} < \infty$, (3.66) holds. □

**Lemma 3.2.2** (p.3714, [115]). *Suppose that there exists $\delta > 0$ such that*

$$\sum_{l=1}^{\infty}|\sigma_l|^\beta\lambda_l^{\beta\delta} < \infty.$$

*Then for all $p \in (0, \beta)$ and $T > 0$*

$$\mathbb{E}\sup_{0\leq t\leq T}|A^\delta z_t|^p \leq C\left(1 + T^{p(1-\delta)}\right)T^{p/\beta} \tag{3.67}$$

**Proof.** It is proved in [115] that for $p > 1$

$$\mathbb{E} \sup_{0 \le t \le T} |A^\delta z_t|^p \le C T^{p/\beta} . \tag{3.68}$$

In order to prove the lemma for the process $z$, we use formula (3.64). Let $Z = z - z^0$. Then (3.64) yields

$$\frac{dZ}{dt} = -AZ - C\left(Z + z^0\right) = -\widehat{A}Z - Cz^0, \quad Z(0) = 0 .$$

Therefore,

$$Z(t) = -\int_0^T e^{-(t-s)\widehat{A}} C z^0(s)\, ds, \quad t \ge 0 .$$

Then, by the properties of analytic semigroups we find that

$$\begin{aligned}
\left|\widehat{A}^\delta Z(t)\right| &\le \int_0^t \left|\widehat{A}^\delta e^{-(t-s)\widehat{A}}\right| \left|Cz^0(s)\right| \, ds \\
&\le \sup_{s \le t} \left|Cz^0(s)\right| \int_0^t \frac{c}{(t-s)^\delta}\, ds \\
&\le c_1 t^{1-\delta} \sup_{s \le t} \left|Cz^0(s)\right| \\
&\le c_1 |C| t^{1-\delta} \sup_{s \le t} \left|z^0(s)\right| .
\end{aligned}$$

Since $\mathbf{C}$ is bounded, we have $D\left(\widehat{A}\right) = D(A)$ by Theorem 2.11 in [86]. Since $A \ge 0$ is selfadjoint, the domains of fractional powers can be identified as the complex interpolation spaces, see Section 1.15.3 of [107]. Therefore, $D\left(A^\delta\right) = D\left(\widehat{A}^\delta\right)$ for every $\gamma \in (0,1)$, which yields the existence of constants, $r_1, r_2$ depending on $\delta$ only, such that

$$r_1 \left|\widehat{A}^\delta x\right| \le \left|A^\delta x\right| \le r_2 \left|\widehat{A}^\gamma x\right|, \quad x \in D\left(A^\gamma\right) .$$

Using (3.68) we find that

$$\mathbb{E} \sup_{t \le T} \left|A^\delta Z(t)\right|^p \le c_1^p r_2^p |C|^p T^{p(1-\delta)} \mathbb{E} \sup_{s \le T} \left|z^0(s)\right|^p < \infty.$$

Now the lemma follows since $z(t) = Z(t) + z^0(t)$.

Finally, for completeness we prove the case $p \in (0,1)$ for the process $z^0$. As (3.67) is proved for $q \in (1,\beta)$ we fix $q \in (1,\beta)$ and then

$$\mathbb{E}\left(\sup_{0 \le t \le T} |A^\delta z_t^0|^q\right) \le C T^{q/\beta}.$$

Using the Hölder inequality (see for instance [62], p.191) one has that, that is

$$\mathbb{E}(|\mathrm{X}|^p \cdot 1) \le (\mathbb{E} X^{pq})^{1/q}.$$

We then have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}|A^\delta z_t^0|^p\right)\\
&= \mathbb{E}\left(\left\{\sup_{0\le t\le T}|A^\delta z_t^0|\right\}^p\right)\\
&\le \mathbb{E}\left(\left\{\sup_{0\le t\le T}|A^\delta z_t^0|\right\}^{pq}\right)^{1/q}\\
&\le \mathbb{E}\left(\left\{\sup_{0\le t\le T}|A^\delta z_t^0|\right\}^{q}\right)^{p/q}\\
&\le (C_1 T^{q/\beta})^{p/q}\\
&= C_1^{p/q}T^{p/\beta}\\
&\le CT^{p/\beta}.
\end{aligned}
$$

□

**Proposition 3.2.3.** *[p110,[90]] Suppose* $\sum_{l\ge 1}\frac{\sigma_l^\beta}{\lambda+\alpha}<\infty$*, then for any* $0<p<\beta$*,* $t\ge 0$*,*

$$
E|z_t^0|^p \le \tilde{c}_p\left(\sum_{l=1}^\infty |\sigma_l|^\beta \frac{1-e^{-\beta(\lambda_l+\alpha)t}}{\beta(\lambda_l+\alpha)}\right)^{p/\beta},
$$

*where* $c_p$ *depends on* $p$ *and* $\beta$*. Moreover, as* $\alpha\to\infty$*,*

$$
\mathbb{E}|z_t^0|^p \to 0
$$

**Proof.** Under same theme of the proof of Lemma 3.2.1, we follow the argument in the proof of Theorem 4.4 in [90] to complete the proof. Let $z_t^0$ be the solution of

$$
dz_t^0 + (A+\alpha I)z_t^0 = GdL(t), \quad z^0(0)=0
$$

which has the expression

$$
\begin{aligned}
z_t^0 &= \int_0^t S(t-s)GdL(s)\\
&= \sum_{l=1}^\infty\left(\int_0^t e^{-(\lambda_l+\alpha)(t-s)}\sigma_l dL_s^l\right)e_l,
\end{aligned}
$$

where we used the notation $S(t)=e^{-t(A+\alpha I)}$. Take a Radamacher sequence $\{r_k\}_{k\ge1}$ in a new probability space $(\Omega',\mathcal{F}',\mathbb{P}')$, that is $(\Omega',\mathcal{F}',\mathbb{P}')$, that is $\{r_l\}_{l\ge1}$ are i.i.d. with $\mathbb{P}(r_l=1)=\mathbb{P}(r_l=-1)=\frac{1}{2}$. By the following Khintchine inequality: for any $p>0$, there exists some $c_p>0$ such that for any arbitrary real sequence $\{c_l\}_{l\in\mathbb{N}}$,

$$\left(\sum_{l\geq 1} c_l^2\right)^{1/2} \leq c_p \left(\mathbb{E}'|\sum_{l\geq 1} r_l c_l|^p\right)^{1/p},$$

where $c_p$ depends only on $p$.
Now fix $\omega \in \Omega$, $t \geq 0$, write

$$\left(\sum_{l\geq 1} |z_l^0(t,\omega)|^2\right)^{1/2} \leq c_p(\mathbb{E}'|\sum_{l\geq 1} r_l z_l^0(t,\omega)|^p)^{1/p}.$$

Then

$$\begin{aligned}\mathbb{E}|z_t^0|^p &= \left(\sum_{l=1}^{\infty} |\int_0^t e^{-(\lambda_l+\alpha)(t-s)}\sigma_l dL_s^l|^2\right)^{\frac{p}{2}} \\ &\leq c_p^p \mathbb{E}\left(\mathbb{E}'|\sum_{l=1}^{\infty} r_l z_l^0(t)|^p\right) = c_p^p \mathbb{E}'\left(\mathbb{E}|\sum_{l=1} r_l z_t^l|^p\right) = c_p^p \mathbb{E}'\left(\mathbb{E}|\sum_{l=1}^{\infty} r_l \int_0^t e^{-(\lambda_l+\alpha)(t-s)}\sigma_l dL_s^l|^p\right).\end{aligned}$$

For any $t \geq 0$, $\kappa \in \mathbb{R}$ using the fact $|r_l| = 1$, $l \geq 1$ and formula (4.7) in [90],

$$\mathbb{E}e^{i\kappa\sum_{l\geq 1} r_l z_l^0(t)} = e^{-|\kappa|^\beta}\sum_{l\geq 1}|\sigma_l|^\beta \int_0^t e^{-\beta(\lambda_l+\alpha)(t-s)}ds.$$

Now we use (3.2) in [90]: If $X$ is a symmetric $\beta$-stable r.v. with distribution $S(\beta,\gamma,0)$ satisfying

$$\mathbb{E}e^{i\kappa X} = e^{-\gamma^\beta|\kappa|^\beta}$$

for some $\beta \in (0,2)$ and any $\kappa \in \mathbb{R}$, then for any $p \in (0,\beta)$, one has

$$\mathbb{E}X^p = C(\beta,p)\gamma^p.$$

Since $\sum_{l\geq 1}\frac{\sigma_l^\beta}{\lambda_l+\alpha} < \infty$, the assertion follows. Furthermore, $\mathbb{E}|z_t|_p^0 \to 0$ as $\alpha \to \infty$. □

Now we present a Lemma that allows us to claim that the solution of SNSE has càdlàg trajectories. The proof follows closely with Lemma 3.3 in [115].

**Lemma 3.2.4.** *Assume that for a certain $\delta \in [0,1)$*

$$\sum_{l=1}^{\infty}|\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty.$$

*Then the process $z$ defined by (3.65) has a version in $D\left([0,\infty]; D\left(A^\delta\right)\right)$*

**Proof.** By Lemma 3.2.2 we have

$$\mathbb{E}\sup_{0\leq t\leq T}|A^\delta z_t|^p < \infty$$

for any $p \in (0, \beta)$. Now, by Theorem 2.2 in [75] $z^0$ has a càdlàg modification[2] in $V$. By representation (3.64) the process $z$ is càdlàg as well and the proof of Lemma is completed. $\square$

We still study the $\gamma$- radonifying, property, that is equation (2.2) introduced in Chapter 2 of the operator $(I - \Delta)^{-s}$.

**Lemma 3.2.5.** *Let $\Delta$ denotes the Laplace-de Rham operator on $\mathbb{S}^2$ and $q \in (1, \infty)$. Then the operator*

$$(-\Delta + 1)^{-s} : H \to L^q(\mathbb{S}^2) \text{ is } \gamma - \text{radonifying iff } s > 1/2.$$

**Proof.** Let us choose and fix $q \in (1, \infty)$. Let us recall that all the distinct eigenvalues of $-\Delta + 1$ are $\lambda_l + 1 = l(l+1) + 1$, $l = 0, 1, \cdots$, and the correspnding eigenfunctions are given by the divergence free vector spherical harmonic $\mathbf{Y}_{l,m}$ for $|m| \le l$, $l \in \mathbb{N}$ (p.216 [111]). Let us recall also the addition theorem for vector spherical harmonic

$$\sum_{|m|\le l} |\mathbf{Y}_{l,m}(x)|^2 = \frac{2l+1}{4\pi} P_l(1), \quad x \in \mathbb{S}^2.$$

Now, it is clear that

$$\begin{aligned}
&\mathbb{E}\left|\sum_{l=0}(l(l+1)+1)^{-s}\sum_{|m|\le l}\mathbf{Y}_{l,m}(x)\right|^q_{\mathbb{L}^q} \\
&= \int_{\mathbb{S}^2}\mathbb{E}\left|\sum_{l=0}(l(l+1)+1)^{-s}\sum_{|m|\le l}\mathbf{Y}_{l,m}(x)\right|^q dS(x) \\
&\simeq c_q\int_{\mathbb{S}^2}\left(\sum_{l=0}(l(l+1)+1)^{-2s}\frac{2l+1}{4\pi}P_l(1)\right)^{q/2} dS(x)
\end{aligned}$$

converges if and only if $s > \frac{1}{2}$. $\square$

Let $X = \mathbb{L}^4(\mathbb{S}^2) \cap H$ be the Banach space endowed with the norm

$$|x|_X = |x|_H + |x|_{\mathbb{L}^4(\mathbb{S}^2)}.$$

It follows from Lemma 3.2.5 that the operator

$$A^{-s} : H \to X \text{ is } \gamma - \text{radonifying iff } s > 1/2. \tag{3.69}$$

We need the OU process to take value in $X$, to this end, we need the following assumption.

[2]Modification with càdlàg path.

**Definition 3.2.6.** Let $K$ and $X$ be separable Banach spaces and let $\gamma_K$ be the canonical cylindrical (finitely additive) Gaussian measure on $K$. A bounded linear operator $U : K \to X$ is said to be $\gamma$-radonifying iff $U(\gamma_K)$ is a Borel Gaussian measure on $X$.
One has to choose $X$ wisely, so that $U : K \to X$ is $\gamma$-radonifying (in checking validifty of subordinator condition (2.12) The following is our standing assumption.

**Assumption 1** A continuously embedded Hilbert space $K \subset H \cap \mathbb{L}^4$ is such that for any $\delta \in (0, 1/2)$,

$$A^{-\delta} : K \to H \cap \mathbb{L}^4 \quad \text{is } \gamma\text{-radonifying.} \tag{3.70}$$

It follows from (3.69) that $K = D(A^s)$ for some $s > 0$, then assumption 1 is satisfied.

**Remark.** Under the above assumption, we have the facts $K \subset H$ and Banach space $X$ is taken as $H \cap L^4$. In fact, space $K := Q^{1/2}(W)$ is the RKHS of noise $W(t)$ on $H \cap \mathbb{L}^4$ with inner product $\langle \cdot, \cdot \rangle_K = \langle Q^{-1/2}x, Q^{-1/2}y \rangle_W$, $x, y \in K$. The notation $Q$ denotes the covariance of the noise $W$.
Note: The parameters used in Lemma 3.2.5 and Assumption 1 are independent. In the first case, we start with the whole space, a smaller exponent is required to map onto $H \cap \mathbb{L}^4(\mathbb{S}^2)$, so the assumption $s > 1/2$ justifies. While in Assumption 1, we start with a smaller space, a bigger exponent is required to map onto $H \cap \mathbb{L}^4(\mathbb{S}^2)$, so $\delta \in (0, 1/2)$.

**Corollary 1.** *In the framework of Proposition 3.1.6, let us additionally assume that there exists a separable Hilbert space $K \subset X$ such that the operator $A^{-\delta} : K \to X$ is $\gamma$-radonifying for some $\delta \in (0, \frac{1}{2})$. Then*

$$\int_0^\infty |e^{-tA}|^2_{R(K,X)} dt < \infty.$$

**Proof.** Since $e^{-tA} = A^\delta e^{-tA} A^{-\delta}$, it follows by Neidhardt [81] that

$$|e^{-tA}|_{R(K,X)} \leq |A^\delta e^{-sA}|_{\mathcal{L}(X,X)} |A^{-\delta}|_{R(K,X)},$$

and then Proposition 3.1.6 yields finiteness of the integral. □

Let us recall what one means by $M$-type $p$ Banach space [17]. Suppose $p \in [1, 2]$ is fixed, the Banach space $E$ is called as type $p$, iff there exists a constant $K_p(E) > 0$ such that for any finite sequence of symmetric independent identically distributed r.v. $\xi_1, \cdots, \xi_n : \Omega \to [-1, 1]$, $n \in \mathbb{N}$, and any finite sequence $x_1, \cdots, x_n$ from $E$, satisfying

$$\mathbb{E} \left| \sum_{i=1}^n \xi_i x_i \right|^p \leq K_p(E) \sum_{i=1}^n |x_i|^p.$$

Moreover, a Banach space $E$ is of martingale type $p$ iff there exists $L_p(E) > 0$ such that for any $E$-valued martingale $\{M_n\}_{n=0}^N$ the following holds

$$\sup_{n\leq N} \mathbb{E}|M_n|^p \leq L_p(E) \sum_{n=0}^{N} \mathbb{E}|M_n - M_{n-1}|^p.$$

**Lemma 3.2.7** (Corollary 8.1,[23]). *Assume that $p \in (1,2]$, $X$ is a subordinator Lévy process from the class Sub(p), $E$ is a separable type p Banach space, $U$ is a separable Hilbert space, $E \subset U$ and $W = (W(t), t \geq 0)$ is an $U$-valued Wiener process.*
*Define a $U$-valued Lévy process as*

$$L(t) = W(X(t)), \quad t \geq 0 .$$

*Then the $E$-valued process*

$$z(t) = \int_0^t e^{-(t-s)(\mathrm{A}+\alpha I)} dL(s)$$

*is well defined. Moreover, with probability 1, for all $T > 0$,*

$$\int_0^T |z(t)|_E^p dt < \infty,$$

$$\int_0^T |z(t)|_{L^4}^4 .dt < \infty$$

The following existence and regularity result is a version of the result in [23].

**Theorem 3.2.8.** *Let the process $L$ be defined in the same way as in Lemma 3.2.7. Assume that one of the following conditions is satisfied:*

*(i) $p \in (0,1]$ or*

*(ii) the Banach space $E$ is separable of martingale type $p$ for a certain $p \in (1,2]$.*

*Then the process*

$$z_\alpha(t) = \int_{-\infty}^t e^{-(t-s)(\hat{A}+\alpha I)} dL(s) \tag{3.71}$$

*is well defined in $E$ for all $t > 0$. Moreover, if $p \in (1,2]$, then the process $z$ of (3.71) is càdlàg.*

**Proof.** As $S = (S(t), t \geq 0)$ is a $C_0$ semigroup in the separable martingale type $p$-Banach space $E$, there exists a Hilbert space $H$ as the reproducing Kernel Hilbert space of $W(1)$ such that the embedding $i : H \hookrightarrow E$ is $\gamma$-radonifying. The proof of this theorem is a straight application of Theorem 4.1 and 4.4 in [23]. □

In order to obtain well-posedness of the SNSE 3.59, one need some regularity on the noise term. Fortunately, this becomes attainable using Lemma 3.2.7 . In view of this, we construct the driving Lévy noise $L = L(t)$ by subordinating a cylindrcial Wiener process $W$ on a Hilbert space $H$. Let $\{W_t^l, t \geq 0\}$ be a sequence of independent standard one-dimensional Wiener process on some given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cylindrical Wiener process on $H$ is defined by

$$W(t) := \sum_l W_t^l e_l,$$

where $e_l$ is the complete orthonormal system of eigenfunctions on $H$.
For $\beta \in (0, 2)$, let $X(t)$ be an independent symmetric $\beta/2$-stable subordinator, that is, an increasing one dimensional Lévy process with Laplace Transform

$$\mathbb{E}e^{-rX(t)} = e^{-t|r|^{\beta/2}}, \quad r > 0$$

The subordinated cylindrical Wiener process $\{L(t), t \geq 0\}$ on $H$ is defined by

$$L(t) := W(X(t)), \quad t \geq 0.$$

Note in general that $L(t)$ does not belongs to $H$. More precisly, $L(t)$ lives on some larger Hilbert space $U$ with the $\gamma$-radonifying embedding $H \hookrightarrow U$. In this chapter we consider abstract Itô equation in (3.61) (which we restate here) in $H = L^2(\mathbb{S}^2)$ :

$$du(t) + \nu Au(t)dt + B(u(t), u(t))dt + \mathbf{C}u = fdt + GdL(t), \quad u(0) = u_0. \tag{3.72}$$

Write (3.61) into the usual mild form one has

$$u(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)fds + \int_0^t S(t-s)GdL(s). \tag{3.73}$$

where $S(t)$ is an analytic $C_0$ semigroup $(e^{-t\hat{A}})$ generated by $\hat{A} = \nu A + \mathbf{C}$, where $A$ is the Stokes operator in $H$. Note that $\hat{A}$ is a strictly positive selfadjoint operator in $H$ (that is $A : D(A) \subset H \to H$, $\hat{A} = \hat{A}^* > 0$, $\langle Av, v\rangle \geq \gamma|v|^2$ for any $v \in D(A)$ for some $\gamma > 0$ and $v \neq 0$). The operator $G : H \to H$ is a bounded linear operator. For a fixed $\alpha > 0$ we introduce the process

$$z_\alpha(t) := \int_0^t e^{-(t-s)(\alpha+\widehat{A})}GdL(t)$$

that solves the OU equation

$$dz_\alpha + (\nu A + \mathbf{C} + \alpha)z_\alpha dt = GdL(t), \quad t \geq 0\,. \tag{3.74}$$

Now let $v(t) = u(t) - z_\alpha(t)$. Then

$$\begin{cases} dv(t) + \nu A(u(t) - z_\alpha(t))dt + B(u(t))dt + \mathbf{C}(u - z_\alpha(t))dt - \alpha z_\alpha(t)dt = fdt, \\ v(0) = v_0. \end{cases}$$

The problem becomes

$$\begin{cases} dv(t) + \nu Av(t)dt + B(v(t) + z_\alpha(t))dt + \mathbf{C}v(t)dt - \alpha z_\alpha(t)dt = fdt, \\ v(0) = v_0. \end{cases}$$

Convert into standard form,

$$\begin{cases} \frac{d^+}{dt}v(t) + (\nu A + \mathbf{C})v(t) = f - B(v(t) + z_\alpha(t)) + \alpha z_\alpha(t), \\ v(0) = v_0, \end{cases} \tag{3.75}$$

where $\frac{d^+v}{dt}$ is the right-hand derivative of $v(t)$ at $t$. Solution to equation (3.75) will be understood in the mild sense, that is as a solution to the integral equation

$$v(t) = S(t)v(0) + \int_0^t S(t-s)(f - B(v(s) + z_\alpha(s)) + \alpha z_\alpha(s))ds, \tag{3.76}$$

with $v_0 = u_0 - z_\alpha(0)$. One can easily show that (3.75) and (3.76) are equivalent for $v \in C(0,\infty;V) \cap L^2_{\mathrm{loc}}(0,\infty;D(A))$. More precisely, (3.76) follows from (3.75) via integration. Then (3.75) follows from (3.76) via the usual continuity argument (see Lebesgue Dominated Convergence Theorem in Appendix), namely, differentiation the integral when integrand is continuous.
For brevity, we write $z_\alpha$ as $z$. Let us now explain what is meant by a solution of (3.61).

**Definition 3.2.9.** Suppose that $z \in L^4_{\mathrm{loc}}([0,T];\mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$, $f \in V'$. A weak solution to (3.61) is a function $v \in C([0,T);H) \cap L^2_{\mathrm{loc}}([0,T);V)$ satisfies (3.75) in weak sense for any $\phi \in V$, $T > 0$,

$$\partial_t(v,\phi) = (v_0,\phi) - \nu(v, A\phi) - b(v+z, v+z, \phi) - (\mathbf{C}v,\phi) + (\alpha z + f, \phi). \tag{3.77}$$

Equivalently, (3.75) holds as an equality in $V'$ for a.e. $t \in [0,T]$.
Now if $f \in H$, and the following regularity is satisfied,

$$v \in L^\infty(0,T;V) \cap L^2(0,T;D(A)), \tag{3.78}$$

then the solution becomes strong. More precisely,

**Definition 3.2.10** (Strong solution)**.** Suppose that $z \in L^4_{\mathrm{loc}}([0,T);\mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in V$, $f \in H$. We say that $u$ is a *strong solution* of the stochastic Navier-Stokes equations (3.61) on the time interval $[0,T]$ if $u$ is a weak solution of (3.61) and in addition

$$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A)). \tag{3.79}$$

### 3.2.2 A summary of main theorems

In this subsetcion we state the main theorems proved in this chapter are the following.

**Theorem 3.2.11.** *Suppose that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$ and $f \in V'$. Then there exists a unique solution $v$ of equation (3.75). In particular, if*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty\,,$$

*then $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$.*

Next, we show the weak solution depends continuously on initial data, noise and forcing terms.

**Theorem 3.2.12.** *Assume that,*

$$u_n^0 \to u \quad in \quad H,$$

*and for some $T > 0$,*

$$z_n \to z \quad in \quad L^4([0,T];\mathbb{L}^4(\mathbb{S}^2) \cap H) \qquad f_n \to f \quad in \quad L^2(0,T;V'). \tag{3.80}$$

*Let us denote by $v(t,z)u_0$ the solution of (3.75) and by $v(t,z_n)u_n^0$ the solution of (3.75) with $z, f, u_0$ being replaced by $z_n, f_n, u_n^0$. Then*

$$v(\cdot,z_n)u_n^0 \to v(\cdot,z)u_0 \quad in \quad C([0,T];H) \cap L^2(0,T;V).$$

**Theorem 3.2.13.** *Suppose that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$ and $f \in V'$. Then there exists $\mathbb{P}$-a.s. a unique solution $u \in D([0,\infty);H) \cap L^2_{loc}([0,\infty);V)$ of equation (3.61). In particular, if*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty\,,$$

*then $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$.*

Analogously to Theorem 3.2.12, the (càdlàg in time) solution to the SNSE depends continuously on initial data, noise and forcing terms.

**Theorem 3.2.14.** *Assume that,*

$$u_n^0 \to u \quad in \quad H$$

*and for some $T > 0$,*

$$z_n \to z \quad in \quad L^4([0,T];\mathbb{L}^4(\mathbb{S}^2)\cap H) \qquad f_n \to f \quad in \quad L^2(0,T;V'). \tag{3.81}$$

*Let us denote by $u(t,z)u_0$ the solution of (3.75) and by $u(t,z_n)u_n^0$ the solution of (3.75) with $z, f, u_0$ being replaced by $z_n, f_n, u_n^0$. Then*

$$u(\cdot,z_n)u_n^0 \to u(\cdot,z)u_0 \quad in \quad D([0,T];H)\cap L^2(0,T;V).$$

*In particular, $u(T,z_n)u_n^0 \to u(T,z_n)u_0$ in $H$.*

Moreover, the weak solution is found to be strong indeed.

**Theorem 3.2.15.** *Assume that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2)\cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists unique solution of (3.76) in the space $C(0,T;H)\cap L^2(0,T;V)$ which belongs to $C(h,T;V)\cap L^2_{loc}(h,T;D(A))$ for all $h > 0$ and $T > 0$. Moreover, if $v_0 \in V$, then $v \in C(0,T;V)\cap L^2_{loc}(0,T;D(A))$ for all $T > 0$. In particular, $v(T,z_n)u_n^0 \to v(T,z_n)u_0$ in $H$. Moreover, if*

$$\sum_{l=1}^{\infty} |\sigma_l|^{\beta}\lambda_l^{\beta/2} < \infty\,,$$

*then $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2)\cap H)$.*

**Theorem 3.2.16.** *Assume that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2)\cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists $\mathbb{P}$-a.s. unique solution of (3.61) in the space $D(0,T;H)\cap L^2(0,T;V)$. which belongs to $D(\epsilon,T;V)\cap L^2_{loc}(\epsilon,T;D(A))$ for all $\epsilon > 0$. and $T > 0$. Moreover, if $v_0 \in V$, then $u \in D(0,T;V)\cap L^2_{loc}(0,T;D(A))$ for all $T > 0$, $\omega \in \Omega$. Moreover, if*

$$\sum_{l=1}^{\infty} |\sigma_l|^{\beta}\lambda_l^{\beta/2} < \infty\,,$$

*then $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2)\cap H)$.*

**Theorem 3.2.17.** *Assume additionally, that there exists $m > 1$ such that $\sigma_l = 0$ for all $l \geq m$. Then the solution $u$ to (3.61) admits at least one invariant measure.*

## 3.3 Weak solutions

In standard PDE theory, it is often convenient to recast the original problem in an appropriate 'weak form', and then to seek for a solution of the transformed equation, that is, the so-called 'weak solution'. Roughly speaking, the weak formulation allows one to exploit techniques that are not available in its classical form, and lead to a weak solution. Ideally, once the existence of a weak solution is found, then it would be possible to show that the weak solution is indeed smooth and hence becomes a classical solution. For our 2D stochastic Navier-Stokes equation, we are able to prove the existence of weak solutions for all positive times as well as their smoothness (or in other words, the so-called 'strong solution'.) .

### 3.3.1 Motivation

Let us write (3.75) in a slightly different way as

$$\begin{cases} \partial_t v(t) + (\nu A + \mathbf{C})v(t) = f - B(v(t) + z_\alpha(t)) + \alpha z_\alpha(t), \\ v(0) = v_0 \end{cases}$$

Let $F = -B(z) + \alpha z + f$ and rearrange, one gets

$$\begin{cases} \partial_t v(t) + \nu A v(t) = -\mathbf{C}v(t) - B(v) - B(v,z) - B(z,v) + F, \\ v(0) = v_0 \end{cases} \tag{3.82}$$

Multiply (3.82) with $v$, then using

$$(Av, v) = |\nabla v|^2 = |v|_V, (\mathbf{C}v, v) = 0, B(z, v, v) = 0,$$

one gets

$$\frac{1}{2}\partial_t |v|^2 + \nu |Dv|^2 = -B(v, z, v) + \langle F(t), v\rangle, \quad v(0) = v_0.$$

Now we know,

$$B(z) \in V', \quad |z|_{V'} \leq C|z|_4,$$

so $z$ is in $V'$ and so $F \in V'$. So

$$\langle F(t), v\rangle = \frac{1}{\nu}|F(t)|^2_{V'} + \frac{\nu}{4}|Dv|^2, \tag{3.83}$$

$$|b(v, v, z)| = \frac{\nu}{4}|v(t)|^2_V + \frac{C}{\nu^3}|v|^2|z|^2_V, \tag{3.84}$$

$$\frac{1}{2}\partial_t|v|^2 + \nu|Dv|^2 \leq \frac{\nu}{4}|Dv|^2 + \frac{C}{\nu}|v|^2|Dz|^2 + \frac{1}{\nu}|F(t)|^2_{V'} + \frac{\nu}{4}|Dv|^2,$$

$$\partial_t|v|^2 + \nu|Dv|^2 \leq \frac{2C}{\nu}|Dz|^2|v|^2 + \frac{2}{\nu}|F(t)|^2_{V'}. \tag{3.85}$$

Then apply Gronwall lemma to

$$\partial_t|v|^2 \leq \frac{C}{\nu}|Dz|^2|v|^2 + \frac{2}{\nu}|F(t)|^2_{V'}.$$

One has

$$\sup_{t\in[0,T]}|v|^2 \leq |v(0)|^2 \exp\left(\frac{1}{2\nu}\int_0^T |D(z(\tau))|^2 d\tau\right) + \int_0^T \frac{2}{\nu}|F(t)|^2_{V'}\left(\exp\int_s^T \frac{1}{2\nu}|Dz(\tau)|^2 d\tau\right)dt.$$

This inequality is a promising a priori estimate indicates that $T = \infty$. Fix $T > 0$, denoting

$$\Psi_T(z) = \exp\left(\frac{1}{2\nu}\int_0^T |D(z(\tau))|^2 d\tau\right), \quad C_F = \int_0^T \frac{2}{\nu}|F(t)|^2_{V'}\left(\exp\int_s^T \frac{1}{2\nu}|Dz(\tau)|^2 d\tau\right)dt.$$

The above inequality becomes

$$\sup_{t\in[0,T]}|v|^2 \leq |v(0)|^2\Psi_T(z) + C_F < \infty, \quad t \in [0,T], \tag{3.86}$$

which implies

$$v \in L^\infty(0,T;H), \tag{3.87}$$

then integrate in time (3.85) from 0 to $T$, one gets

$$|v(T)|^2 + \nu\int_0^T |Dv(s)|^2 ds \leq |v_0|^2 + \frac{1}{2\nu}\int_0^T |Dz(s)|^2\left(|v(0)|^2\psi_T(z) + C_F\right)dt + \frac{2}{\nu}\int_0^T |F(t)|^2 dt, \tag{3.88}$$

which implies

$$v \in L^2(0,T;H). \tag{3.89}$$

Therefore,

$$v \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{3.90}$$

Indeed, this essentially yields the definition of weak solution. Furthermore, since $v$ solves (3.75), $v \in L^2(0,T;V)$ and $A : V \mapsto V'$ is a bounded linear operator, $Av \in L^2(0,T;V')$. Since $z \in L^4_{\text{loc}}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$ which can be continuously embedded into $H^{-1}$ since $H^1$ is continuously embedded into $L^4$. It then follows that all terms $-B(z) + \alpha z + f \in L^2(0,T;V')$, $B(v)$, $B(v,z)$, $B(z,v)$ belongs to $L^2(0,T;V')$. Hence $\partial_t v \in L^2(0,T;V')$. Thus, it follows from a classical fact $\partial_t|u(t)|^2 = 2\langle\partial_t u(t), u(t)\rangle$ (see Lion and Magenes p.238 1.2 [72]) that $v \in C([0,T];H)$.

### 3.3.2 Existence of Weak solutions via Galerkin approximation

Our aim in this subsection is to prove the existence part of Theorem 3.2.11. *First*, we construct approximate solutions and deduce local existence and uniqueness of the solutions of the Galerkin equations of SNSE. (For a comprehensive overview of Galerkin methods on spheres, we refer readers to [69].) *Next*, we obtain uniform a prior estimates on the solutions $v_L$ and hence show that they exist globally in time. *Last* but not least, we extract a convergent subsequence and pass to the limit in the equation.

**Definition 3.3.1.** The $L$-order Galerkin equations for the SNSE (3.82) is given by

$$\begin{cases} \partial_t v_L(t) = P_L[-\nu A v_L - B(v) - B(v,z) - B(z,v) - \mathbf{C}v(t) + F], \\ v(0) = P_L v_0, \end{cases} \tag{3.91}$$

where $F = -B(z) + \alpha z + f$ and operators $A$, $B$ and $\mathbf{C}$ are defined respectively in (3.34), (3.58) and (3.38). We call $v_L$ the Galerkin approximations.

#### 3.3.2.1 Local existence and uniquness of $v_L$

For any $L \in \mathbb{N}$ denote

$$H_L = \text{linspan}\{\mathbf{Z}_{l,m} : l = 1, \cdots, L; |m| \leq l\},$$

as the linear space spanned by the first $L$ eigenfunctions in an orthonormal basis $\{\mathbf{Z}_{l,m} : l = 1, \cdots, L; |m| \leq l\}$ of $H$, which may be assumed to be the orthogonal in $V$. In other words, $H_L$ is the $L$-dimensional subspace of $V$ and $P_L$ is the orthogonal projection from $H$ onto $H_L$ defined as

$$P_L v = \sum_{l=1}^{L} \sum_{m=-l}^{l} (v, \mathbf{Z}_{l,m}) \mathbf{Z}_{l,m}.$$

We replace (3.82) by the Galerkin approximations (3.91), where $F = -B(z) + \alpha z + f$. In view of (3.53), $B(z)$ belongs to the dual space $V'$ and so $F \in L^2(0, T; V')$.
We notice that (3.91) is an Ordinary Differential Equations (ODE) in $H_L$, hence the existence and uniqueness of solution $v_L$ of (3.91) defined on $[0, T_L)$ follows from standard theory of ODE. Since the right-hand side has a bilinear form, it is not clear if $v_L$ can be defined globally or it could blow up at some time $T_L < \infty$. We will show in the next subsubection that the $H$ norm of the solution stay finite as $t \to T_L$, which implies the solution indeed exists globally in time.

#### 3.3.2.2 Uniform a-priori estimates on the solutions $v_L$

From the last subsubsection we already know that $v_L$ exists on some time interval $[0, T_L)$. Now we want to send $L$ to infinity and to show a subsequence of the solution $v_L$ of the approximate problem converges to a weak solution to (3.82). For this, we need some uniform estimates. Take the inner product of (3.91) in $H$ with $v_L(t)$ we obtain

$$(\partial_t v_L(t), v_L(t)) = -\nu(P_L A v_L, v_L) - (P_L B(v_L), v_L) - (P_L B(v_L, z), v_L) - (P_L B(z, v_L), v_L) - (P_L \mathbf{C} v_L, v_L) + \langle F, v_L\rangle.$$

We notice that

$$(\partial_t v_L(t), v_L(t)) = \frac{1}{2}\frac{d^+}{dt}|v_L(t)|^2,$$
$$-\nu(P_L A v_L, v_L) = -\nu(A v_L, v_L) = -\nu|v_L|_V^2,$$

and by (3.48),

$$(P_L B(v_L), v_L) = (B(v_L), v_L) = b(v_L, v_L, v_L) = 0, \quad (P_L B(z, v_L), v_L) = b(z, v_L, v_L) = 0,$$

and by (3.38),

$$(P_L \mathbf{C} v_L, v_L) = (\mathbf{C} v_L, v_L) = 0.$$

Therefore, for any $t > 0$ we have

$$\frac{1}{2}\frac{d^+}{dt}|v_L(t)|^2 = -\nu|v_L(t)|_V^2 - b(v_L(t), v_L(t), v_L(t)) + \langle F(t), v_L(t)\rangle \quad t \in [0, T_L).$$

Using (3.48) and (3.56) and the Young inequality ($ab \le \frac{a^p}{p} + \frac{b^q}{q}$ with $p = 4$, $q = 4/3$), we have

$$\begin{aligned} |b(v_L, v_L, z)| &\le C|v_L|_{\mathbb{L}^4(\mathbb{S}^2)}|v_L|_V|z|_{\mathbb{L}^4(\mathbb{S}^2)} \\ &\le C|v_L|^{1/2}|v_L|_V^{3/2}|z|_{\mathbb{L}^2(\mathbb{S}^2)} \\ &\le C|v_L|^{1/2}|v_L|_V^{3/2}|z|_V \\ &\le \frac{C}{\nu^3}|v_L|^2|z|_V^4 + \frac{\nu}{4}|v_L|_V^2. \end{aligned}$$

We also have

$$\langle F(t), v_L\rangle \le |F(t)|_{V'}|v_L|_V \le \frac{1}{\nu}|F(t)|_{V'}^2 + \frac{\nu}{4}|v_L|_V^2.$$

Hence we obtain

$$\partial_t|v_L(t)|^2 + \nu|v_L|_V^2 \le \frac{C}{\nu^3}|v_L|^2|z|_V^4 + \frac{2}{\nu}|F(t)|_{V'}^2, \quad t \in [0, T_L). \tag{3.92}$$

Invoking Gronwall Lemma (see appendix), one has

$$|v_L(t)|^2 \le |v(0)|^2 \exp\left(\frac{C}{\nu^3}\int_0^t |z(\tau)|_V^4 d\tau\right) + \int_0^t \frac{2}{\nu}|F(s)|_{V'}^2 \exp\left(\frac{C}{\nu^3}\int_s^t |z(\tau)|_V^4 d\tau\right) ds, \quad t \in [0, T_L).$$

It follows that $v_L$ does not blow up in finite time and so $T_L = \infty$.

Let us fix $T > 0$. Denoting

$$\psi_T(z) = \exp\left(\frac{C}{\nu^3}\int_0^T |z(\tau)|_V^4 d\tau\right) < \infty, \quad C_F = \int_0^T \frac{2}{\nu}|F(t)|_{V'}^2 \exp\left(\frac{C}{\nu^3}\int_s^T \frac{1}{2\nu}|z(\tau)|^2 d\tau\right) dt$$

We find that

$$\sup_{t\in[0,T)} |v_L(t)|^2 \leq |v_L(0)|^2\psi_T(z) + C_F \leq |v(0)|^2\psi_T(z) + C_F < \infty \quad t \in [0, T) \tag{3.93}$$

which implies that $\{v_L : L \in \mathbb{N}\}$ is bounded uniformly (in $L$) in the norm of $L^\infty(0, T; H)$. Next we integrate in time (3.92) from 0 to $T$ and then using (3.93) to obtain

$$|v_L(T)|^2 + \nu\int_0^T |v_L(t)|_V^2 dt + \frac{C}{\nu^3}\int_0^T |z(t)|_V^2|v_L(t)|^2 dt + \frac{2}{\nu}\int_0^T |F(t)|_{V'}^2 dt.$$

We will now pass to the limits by sending $L$ to infinity, to build a weak solution of our original problem (3.82). For this we need some convergence results. Notice that the above inequality implies that

$$\text{the sequence } \{v_L : L \in \mathbb{N}\} \text{ is bounded uniformly in } L^2(0, T; V) \tag{3.94}$$

Therefore we have shown that $v_L$ is uniformly bounded in $L$ in the norm of $L^\infty(0, T; H) \cap L^2(0, T; V)$. These uniform bounds imply that $\{v_L\}$ has a subsequence that converges weakly in $L^2(0, T; V)$ and weakly* in $L^\infty(0, T; H)$. Then by the Banach-Alaogu theorem (see Appendix), one can extract a subsequence $\{v_{L_k} \subset v_L\}$ and some limit function $v \in L^2(0, T; V)$ such that

$$\begin{cases} v_L \rightharpoonup v, & \text{weakly in } L^2(0, T; V), \\ v_L \rightharpoonup v, & \text{weakly* in } L^\infty(0, T; H). \end{cases} \tag{3.95}$$

Now we need to show

$$v_L \to v \quad \text{strongly} \quad \text{in} \quad L^2(0, T; H), \tag{3.96}$$

and this strong convergence result allows us to choose $v_L$ such that $v_L \to v$ in $L^2(\mathbb{S}^2)$ for all $t \geq 0$ The crux to prove (3.96) is a compactness theorem which involves fractional derivatives. Now, Let us assume that $X_0 \subset X \subset X_1$ are Hilbert spaces with the injection being continuous and the injection of $X_0$ into $X$ is compact. If $v$ is a function from $\mathbb{R}$ to $X_1$, let us denote $\hat{v}$ the Fourier Transform as

$$\hat{v}(\tau) = \int_{\mathbb{R}} e^{-2\pi i t\tau} v(t) dt, \quad \tau \in \mathbb{R}. \tag{3.97}$$

The fractional derivative in $t$ of order $\gamma$ of $v$ is the Fourier transform of the $X_1$-valued function $\{\mathbb{R} \ni \tau \mapsto (2i\pi\tau)^\gamma \hat{v}(\tau)\}$:

$$\widehat{D_t^\gamma} v(\tau) = (2i\pi\tau)^\gamma \hat{v}(\tau), \quad \tau \in \mathbb{R}.$$

The definition makes sense, observe that the first derivative of (3.97) via integration by part is obtained as,

$$\begin{aligned}\widehat{D_t}v(\tau) &= \int_{\mathbb{R}} e^{-2i\pi\tau t} v'(t)dt \\ &= e^{-2\pi i\tau t}v(t)|_{-\infty}^{\infty} - \left(-2\pi i\tau \int_{\mathbb{R}} v(t)e^{-2\pi i\tau t}dt\right)\end{aligned}$$

Since $|v(t)| \to 0$ as $|t| \to \infty$, the first term vanishes, and so

$$\widehat{D_t}v(\tau) = 2\pi i\tau\hat{v}_L(t).$$

For a given $\gamma > 0$, we define the space

$$\mathcal{H}^{\gamma}(\mathbb{R}; X_0, X_1) = \{v \in L^2(\mathbb{R}; X_0) : D_t^{\gamma} v \in L^2(\mathbb{R}; X_1)\}, \tag{3.98}$$

as a Hilbert space equipped with the norm

$$\|v\|_{\mathcal{H}^{\gamma}}(\mathbb{R}; X_0, X_1) = (\|v\|^2_{L^2(\mathbb{R};X_0)} + \||\tau|^{\gamma}\hat{v}\|_{L^2(\mathbb{R};X_1)})^{1/2}.$$

For a given set $K \subset \mathbb{R}$, the subspace $\mathcal{H}_K^{\gamma,2}$ of $\mathcal{H}^{\gamma} = \mathcal{H}^{\gamma,2}(\mathbb{R}; X_0, X_1)$ is defined by

$$\mathcal{H}_K^{\gamma}(\mathbb{R}; X_0, X_1) = \{u \in \mathcal{H}^{\gamma}(\mathbb{R}; X_0, X_1), \text{spt } u \subset K\}. \tag{3.99}$$

**Theorem 3.3.2** (Chapter III, Theorem 2.2 [105]). *Suppose that $X_0 \subset X \subset X_1$ is a Gelfand triple of Hilbert spaces and the injection of $X_0$ into $X$ is compact. Then for any bounded set $K \subset \mathbb{R}$ and $\gamma > 0$, the injection of $\mathcal{H}_K^{\gamma}(\mathbb{R}; X_0, X_1)$ into $L^2(\mathbb{R}; X)$ is compact.*

To apply this compactness theorem, one first need to identity bounded set. For this let

$$\tilde{v}_L = 1_{(0,T)}v_L,$$

and Let Fourier Transform in the time variable of $\tilde{v}_L$ denotes by $\hat{v}_L$. We would like to show that

$$\int_{\mathbb{R}} |\tau|^{2\gamma}|\hat{v}_L(\tau)|^2 dt < \infty. \tag{3.100}$$

Observe that (3.91) can be written as

$$\frac{d^+}{dt}\tilde{v}_L = \tilde{f}_L + v_L(0)\delta_0 - v_L(T)\delta_T, \tag{3.101}$$

where $\delta_0$ and $\delta_T$ are respectively the Dirac distributions at 0 and $T$ and

$$\begin{aligned}f_L &= F - \nu A v_L - B v_L - B(v_L, z) - B(z, v_L) - \mathbf{C}v_L, \\ \hat{f}_L &= 1_{(0,T)}f_L.\end{aligned}$$

Apply the Fourier Transform to (3.101) (with respect to the time variable $t$) we obtain

$$\widehat{D_t}v(\tau) = (2i\pi\tau)\hat{v}(\tau) = \hat{f}_L(\tau) + v_L(0) - v_L(T)\exp(-2i\pi T\tau), \quad \tau \in \mathbb{R}, \tag{3.102}$$

where $\hat{v}_L$ and $\hat{f}_L$ are the Fourier Transform of $\tilde{v}_L$ and $\tilde{f}_L$ respectively. Multiply this equation with the Fourier Transform of $v_L$ one obtain for each $\tau \in \mathbb{R}$ that

$$2i\pi\tau|\hat{v}_L(\tau)|^2 = (\hat{f}_L(\tau), \hat{v}_L(\tau)) + (v_L(0), \hat{v}_L(\tau)) - (v_L(T), \hat{v}_L(\tau))\exp(-2i\pi T\tau). \tag{3.103}$$

From the Parseval equality, (3.38) and (3.48), one has

$$(\hat{f}_L, \hat{v}_L) = (f_L, v_L) = \langle F, v_L\rangle - \nu(Av_L, v_L) - b(v_L, z, v_L). \tag{3.104}$$

Therefore, via Cauchy Schwartz and (3.84), we have

$$|(f_L, v_L)| \le |F|_{V'}|v_L|_V + \nu|v_L|_V^2 + \frac{C}{\nu^3}|v_L|^2|z|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{C}{\nu^3}|v_L|^2|z|_4^4 + \frac{\nu}{4}|v|_V^2, \tag{3.105}$$

$$\sup_{v\in V, |v|_V\neq 0}\frac{|(f,v)|}{|v|_V} = |f_L|_{V'} \le |F|_{V'} + \nu|v|_V + \frac{C}{\nu^3}|v_L||z|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{\nu}{4}|v_L|_V^2.$$

Now due to (3.94), $|v_L| \le C_1$, then integrate over time we conclude

$$\int_0^T |f_L|_{V'}dt \le \int_0^T\left(|F|_{V'} + \frac{5\nu}{4}|v|_V + \frac{C_1}{\nu^3}|z|^4_{\mathbb{L}^4(\mathbb{S}^2)}\right)dt \tag{3.106}$$

and this stays bounded (w.r.t. $L$) as $F \in L^2(0,T;V')$, $z \in L^4_{\text{loc}}([0,\infty);\mathbb{L}^4(\mathbb{S}^2))$ remains in a bounded set of $L^2(0,T;V)$. Hence, there exists a $C > 0$ such that

$$\sup_{L\in\mathbb{N}}\sup_{\tau\in\mathbb{R}}|\hat{f}_L(\tau)|_{V'} \le C. \tag{3.107}$$

Now observe from (3.103) that

$$\begin{aligned}2i\pi\tau|\hat{v}_L(\tau)|^2 &= (\hat{f}_L(\tau), \hat{v}_L(\tau)) + (v_L(0), \hat{v}_L(\tau)) - (v_L(T), \hat{v}_L(\tau))\exp(-2i\pi T\tau)\\ &\le |(\hat{f}_L(\tau), \hat{v}_L(\tau))| + v_L(0)|\hat{v}_L(\tau)| + v_L(T)|\hat{v}_L(\tau)|e^{-2i\pi T\tau}.\end{aligned}$$

Then from (3.94), we see

$$|v_L(0)| \le c_1, \quad |v_L(T)| \le c_1. \tag{3.108}$$

Combined with (3.107), one deduces that

$$|\tau||\hat{v}_L|^2 \le c_2|\hat{v}_L|_V + c_3|\hat{v}_L| \le c_4|\hat{v}_L|_V. \tag{3.109}$$

Let us fix $\gamma \in (0, 1/4)$. Observe that

$$|\tau|^{2\gamma} \le C(\gamma)(1+|\gamma|)/(1+|\tau|^{1-2\gamma}) \quad \forall\ \tau \in \mathbb{R}, \tag{3.110}$$

we infer that

$$\int_{\mathbb{R}} |\tau|^{2\gamma} \leq C(\gamma) \int_{\mathbb{R}} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} |\hat{v}_L(\tau)|_V^2 d\tau$$
$$\leq c_5 \int_{\mathbb{R}} \frac{|\tau||\hat{v}_L(\tau)|_V^2 d\tau}{1+|\tau|^{1-2\gamma}} + c_6 \int_{\mathbb{R}} |\hat{v}_L(\tau)|_V^2 d\tau.$$

In the last step, the first integral is finite since $\gamma < 1/4$.[3] Then base on Parseval inequality, one has $|\hat{v}_L| = |v_L|$ and $|\hat{v}_L|_V = |v_L|_V$ which is bounded according to (3.95). Hence we have shown

$$\{\tilde{v}_L\} \quad \text{is bounded in} \quad \mathcal{H}^{\gamma}(\mathbb{R}; V, H). \tag{3.111}$$

This allows us to apply the compactness theorem involves fractional derivatives.
Since the sphere $\mathbb{S}^2$ is bounded, the embedding $\mathbb{H}^1(\mathbb{S}^2) \hookrightarrow \mathbb{L}^2(\mathbb{S}^2)$ is compact and by (3.111), the sequence $\{\tilde{v}_L : L \in \mathbb{N}\}$ is bounded in $H^{\gamma}(0, T; \mathbb{H}^1(\mathbb{S}^2), \mathbb{L}^2(\mathbb{S}^2))$. Due to (3.111) and Theorem 3.3.2, we deduce that there exists a subsequence $\{v_{L_k}\}$ such that $\{v_{L_k}\} \to v$ strongly in $L^2(0, T; \mathbb{L}^2(\mathbb{S}^2))$

$$v_L \to v \quad \text{strongly} \quad L^2(0, T; H). \tag{3.112}$$

The convergence result (3.95) and (3.96) enable us to pass to the limit. Now we need to show the limit function indeed satisfies (3.75). Take a $C^1([0, T]; \mathbb{R})$ function $\psi$ with $\psi(T) = 0$. Multiply (3.91) with $\psi(T)\phi$ where $\phi \in H_l$ for some $l \in \mathbb{N}^+$, then integrate by parts, one gets

$$\begin{aligned} -\int_0^T (v_L(t), \psi'(t)\phi) dt = & -\nu \int_0^T (P_L A v_L(t), \psi(t)\phi) dt \\ & - \int_0^T (P_L B(v_L(t)), \psi(t)\phi) dt - \int_0^T P_L B(v_L(t), z, \psi(t)\phi) dt \\ & - \int_0^T P_L B(z, v_L(t), \psi(t)\phi) dt - \int_0^T \langle P_L F(t), \psi(t)\phi \rangle dt + (v_L(0), \psi(0)\phi). \end{aligned} \tag{3.113}$$

Now we aim to pass to the limit of (3.113) when $L \to \infty$. Since $\psi(\cdot)\phi \in L^2(0, T; H)$, $\psi \in C^1(0, T; \mathbb{R})$, then $\psi(\cdot)\phi \in L^2(0, T; V)$, combine with the first part of (3.95), we have

$$\int_0^T \langle v_L(t), \psi'(t) \rangle dt \to \int_0^T \langle v(t), \psi'(t) \rangle dt \quad \text{as} \quad L \to \infty. \tag{3.114}$$

Hence the left hand side of (3.113) converges to $-\int_0^T (v(t), \psi'(t)\phi) dt$.

[3]This integral converges iff. $\int_1^{\infty} x^{2(2\gamma-1)} dx < \infty$. This holds iff. $2(2\gamma - 1) < -1$.

Next, for the linear term, let us take take $l \in L$ so that $H_l \subset H_L$ and $P_L\phi = \phi$. For the first term on the right hand side of (3.95), observe that

$$\begin{aligned}\int_0^T (P_L A v_L(t), \psi(t)\phi) dt &= \int_0^T (A v_L(t), \psi(t) P_L \phi) dt \\ &= \int_0^T (A v_L(t), \psi(t)\phi) dt \\ &= \int_0^T (v_L(t), \psi(t)\phi)_V dt.\end{aligned}$$

Again, since $\psi(\cdot)\phi \in L^2(0, T; V)$, it follows from (3.95) that, as $L \to \infty$,

$$\int_0^T (P_L v_L(t), \psi(t)\phi) dt \to \int_0^T (v(t), \psi(t)\phi)_V dt, \tag{3.115}$$

or

$$\int_0^T (v_L(t) - v(t), \psi(t)\phi)_V dt \to 0. \tag{3.116}$$

**Lemma 3.3.3.** *If $v_m \rightharpoonup v$ in $L^2(0, T; V)$ and strongly in $L^2(0, T; H)$, then for any vector function $u : [0, T] \times \mathbb{S}^2 \to \mathbb{R}^2$ with components in $C^1(\mathbb{S}^2 \times [0, T])$,*

$$\int_0^T b(v_m(t), v_m(t), u(t)) dt \to \int_0^T b(v(t), v(t), u(t)) dt. \tag{3.117}$$

**Proof.**

$$\begin{aligned}\int_0^T b(v_m, v_m, u) dt &= -\int_0^T b(v_m, u, v_m) dt \\ &= -\sum_{i,j=1}^{3} \int_0^T \int_{\mathbb{S}^2} (v_m)_i (D_i u_j)(v_m)_j dx dt.\end{aligned}$$

Now our two assumptions on $v_m$ imply that

$$v_m \to v \quad \text{in} \quad H, \tag{3.118}$$

$$Dv_m \rightharpoonup Dv \quad \text{weakly in} \quad H. \tag{3.119}$$

and which further implies that

$$|v_m(t)|^2 + \sup_{0 \le t \le T} \int_0^T |v_m(t)|_V^2 dt \le C.$$

Hence, there exists a function $g(t)$ for which the term $F_m = (v_m)_i(D_i u_j)(v_m)_j$ is dominated for all $t \in [0,T]$. Now by (3.118),

$$
\begin{aligned}
|v_m| &\le c_1, \quad \text{uniformly in m,}\\
|Du| &\le c_2.
\end{aligned}
$$

Hence $F_m(t) \le g(t) = c_1^2 c_2$. Then by usual continuity argument one has

$$
\begin{aligned}
\lim_{m\to\infty}\int_0^T \sum_{i,j=1}^3 \int_{\mathbb{S}^2} (v_m)_i(D_i u_j)(v_m)_j dxdt &= \int_0^T \lim_{m\to\infty}\sum_{i,j=1}^3 \int_{\mathbb{S}^2} (v_m)_i(D_i u_j)(v_m)_j dxdt\\
&= \int_0^T \sum_{i,j=1}^3 \int_{\mathbb{S}^2} v_i(D_i u_j)v_j dxdt,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T b(v_m, v_m, u)dt &= -\int_0^T b(v_m, u, v_m)dt\\
&= -\sum_{i,j=1}^3 \int_0^T \int_{\mathbb{S}^2} (v_m)_i(D_i u_j)(v_m)_j dxdt\\
&= -\sum \int_0^T \int_{\mathbb{S}^2} v_i(D_i u_j)v_j dxdt\\
&= -\int_0^T b(v,u,v)dt\\
&= \int_0^T b(v,v,u)dt.
\end{aligned}
$$

□

An alternative proof is the following [14].

**Proof.** In view of (3.48), one has $b(v_m, v_m, u) = -b(v_m, u, v_m)$. We also have

$$b(v_m, u, v_m) = -b(v_m, u, v_m).$$

One also have

$$b(v_m, u, v_m) - b(v, u, v) = b(v_m, u, v_m - v) + b(v_m - v, u, v).$$

Using (3.49), combine with the assumption $v_m \to v$ strongly in $L^2(0,T;H)$., one also has

$$|b(v_m, u, v_m - v)| = |b(v_m, v_m - v, u)| \le C|v_m|_V |v_m - v|_V (|\text{curl} u|_{L^\infty(\mathbb{S}^2)} + |u|_{L^\infty(\mathbb{S}^2)}). \tag{3.120}$$

Moreover, invoke the assumption $v_m \to v$ strongly in $L^2(0,T;H)$. again, we conclude that

$$\int_0^T b(v_m, u, v_m - v)dt \to 0. \tag{3.121}$$

Similarly,

$$\int_0^T b(v_m(t) - v(t), u(t), v(t))dt \to 0 \tag{3.122}$$

□

**Lemma 3.3.4.** *Suppose $\{v_m\}$ is bounded in $L^\infty(0,T;H)$, $v \in L^\infty(0,T;H)$, $v_m \rightharpoonup v$ in $L^2(0,T;V)$ and strongly in $L^2(0,T;\mathbb{L}^2_{loc}(\mathbb{S}^2))$. Then for any $w \in L^4(0,T;L^4(\mathbb{S}^2))$,*

$$\int_0^T b(v_m(t), w(t), v_m(t) - v(t))dt \to 0.$$

**Proof.** In view of (3.48), one has $b(v_m, v_m, w) = -b(v_m, w, v_m)$. One also has

$$b(v_m, w, v_m) - b(v, w, v) = b(v_m, w, v_m - v) + b(v_m - v, w, v).$$

Using (3.52), combine with the assumption $v_m \to v$ strongly in $L^2(0,T;H)$., one also has

$$|b(v_m, w, v_m - v)| = |b(v_m, v_m - v, w)| \leq C|v_m|_{L^4(0,T;L^4(\mathbb{S}^2))}|v_m - v|_V|w|_{L^4(0,T;L^4(\mathbb{S}^2))}. \tag{3.123}$$

Moreover, invoke the assumption $v_m \to v$ strongly in $L^2(0,T;\mathbb{L}^2_{loc}(\mathbb{S}^2))$ once again, we conclude that

$$\int_0^T b(v_m(t), w(t), v_m(t) - v(t))dt \to 0. \tag{3.124}$$

Similarly,

$$\int_0^T b(v_m(t) - v(t), v_m(t), v(t))dt \to 0. \tag{3.125}$$

□

Alternatively, one may prove the above Lemma following the proof as in [21]. We detail this here for comparison.

**Proof.** From our assumptions, there exists a constant $C > 0$, such that

$$\sup_{0\leq t\leq T}|v_m(t)| + \sup_{0\leq t\leq T}|v(t)|^{\frac{1}{2}}(\int_0^T |v(t)|^{\frac{1}{2}})\left(\int_0^T |v_m(t)|_V^2 dt\right)^{\frac{3}{4}} + \left(\int_0^T |v_m(t)|^2 dt\right)^{\frac{3}{4}} \leq C.$$

Take $\varepsilon > 0$. Since $w \in L^4(0,T;\mathbb{L}^4(\mathbb{S}^2))$ Hence, via the usual mollification argument, one can find a function $u$ which satisfying the assumptions of previous lemma and such that $(\int_0^T |w(s) - u(s)|^4_{\mathbb{L}^4(\mathbb{S}^2))}ds)^{1/4} < \frac{\varepsilon}{3C^2}$. Hence, by this lemma, we can find $M_\varepsilon \in \mathbb{N}$ such that for for any $m \geq M_\varepsilon$, one has

$$\left|\int_0^T b(v_m(t), v_m(t), u(t)) - \int_0^T b(v(t), v(t), u(t))\right| < \frac{\varepsilon}{3}.$$

Hence, for any $m > M_\varepsilon$,

$$\left|\int_0^T b(v_m(t), v_m(t), w(t))dt - \int_0^T b(v(t), v(t), w(t))dt\right|$$

$$= \left|\int_0^T b(v_m(t), v_m(t), w(t) - u(t))dt + \int_0^T b(v_m(t), v_m(t), u(t))dt - \int_0^T b(v(t), v(t), w(t) - u(t))dt - \int_0^T b(v(t), v(t), u(t))dt\right|$$

$$\leq \left|\int_0^T b(v_m(t), v_m(t), w(t) - u(t))dt\right| + \left|\int_0^T b(v(t), v(t), w(t) - u(t))dt\right|$$

$$+ \left|\int_0^T b(v_m(t), v_m(t), u(t))dt - \int_0^T b(v(t), v(t), u(t))dt\right|$$

Now, by (3.52) and (3.56),

$$\leq \frac{\varepsilon}{3} + \int_0^T |v_m(t)|^{\frac{1}{2}} |v_m(t)|_V^{\frac{3}{2}} |w(t) - u(t)|_{\mathbb{L}^4}^4 dt + \int_0^T |v(t)|^{\frac{1}{2}} |v(t)|_V^{\frac{3}{2}} |w(t) - u(t)|_{\mathbb{L}^4}^4 dt < \varepsilon.$$

□

For the second term of the right hand side of (3.113), that is,

$$\int_0^T (P_L B(v_L(t)), \psi(t)\phi)dt.$$

We apply Lemma 3.3.4 with $w(t, \hat{x}) = \psi(t)\phi(\hat{x})$ for $t \in [0, T]$, $\hat{x} \in \mathbb{S}^2$. Since $P_L$ is self-adjoint in $H$ and $(P_L B(v_L), \psi(t)\phi) = (B(v_L), P_L\psi(t)\phi) = (B(v_L), \psi(t)\phi) = b(v_L, v_L, \psi(t)\phi)$, one obtains the following convergence:

$$\int_0^T (P_L B(v_L(t)), \psi(t)\phi)dt = \int_0^T b(v_L, v_L, \psi(t)\phi)dt \to \int_0^T b(v(t), v(t), \psi(t)\phi)dt.$$

Consider the third term on the the right hand side of (3.113),since

$$\int_0^T P_L B(v_L(t), z, \phi(t)\phi)dt = \int_0^T (B(v_L, z), \psi(t)P_L\phi)dt$$

$$= \int_0^T (B(v_L, z), \psi(t)\phi)dt = \int_0^T b(v_L, z, \psi(t)\phi)dt.$$

Using (3.48) and (3.49) we obtain,

$$\left|\int_0^T (P_L B(v_L, z), \psi(t)\phi)dt - \int_0^T b(v, z, \psi(t)\phi)dt\right|$$

$$= \left|\int_0^T b(v_L(t) - v(t), z(t), \psi(t)\phi)dt\right| = \left|\int_0^T b(v_L(t) - v(t), \psi(t)\phi, z)dt\right|$$

$$\leq \int_0^T |b(v_L(t) - v(t), \psi(t)\phi, z)|dt$$

$$\leq C \int_0^T |v_L(t) - v(t)||z|(|\psi(t)\mathrm{curl}\phi|_{L^\infty(\mathbb{S}^2)} + |\psi(t)\phi|_{L^\infty(\mathbb{S}^2)}).$$

Since $v_L \to v$ strongly in $L^2(0,T;H)$ and $z \in L^4([0,T];\mathbb{L}^4(\mathbb{S}^2)\cap H)$ we infer that the last integral converges to 0 as $L \to \infty$. Hence,

$$\int_0^T (P_L B(v_L, z), \psi(t)\phi) dt - \int_0^T b(v, z, \psi(t)\phi) dt \to 0.$$

Similarly,

$$\int_0^T (P_L B(z, v_L), \psi(t)\phi) dt - \int_0^T b(z, v(t), \psi(t)\phi) dt \to 0.$$

For the fifth term on the rhs of (3.113), we have

$$\int_0^T \langle P_L F, \psi(t)\phi\rangle dt = \int_0^T \langle F, \psi(t) P_L \phi\rangle dt = \int_0^T \langle F, \psi(t)\phi\rangle dt.$$

Now we recall (3.95) to find upon passing to the weak limit of (3.113) that

$$\begin{aligned} -\int_0^T (v(t), \psi'(t)\phi) dt = -\nu \int_0^T (Av(t), \psi(t)\phi) dt - \int_0^T (B(v(t)), \psi(t)\phi) dt - \int_0^T (B(v(t), z, \psi(t)\phi) \\ - \int_0^T (B(z, v(t)), \psi(t)\phi) dt - \int_0^T \langle F(t), \psi(t)\phi\rangle dt + (v_0, \psi(0)\phi). \end{aligned} \tag{3.126}$$

This equality holds for any $\phi \in V$ and any $\psi \in C_0^1([0,T)$. Hence, $v$ solves problem (3.77) and so it satisfies (3.75).

To infer $v$ indeed satisfies (3.75) one also need to show $v(0) = v_0$. For this, let us take an arbitrary function $\phi \in V$ and $\psi \in C_0^1([0,T))$. Multiply (3.75) by $\psi(t)\phi$ then integrate by parts, one gets

$$\begin{aligned} -\int_0^T (v(t), \psi'(t)\phi) dt = -\nu \int_0^T (Av(t), \psi(t)\phi) dt - \int_0^T (B(v(t)), \psi(t)\phi) dt - \int_0^T (B(v(t), z, \psi(t)\phi) \\ - \int_0^T (B(z, v(t)), \psi(t)\phi) dt - \int_0^T \langle F(t), \psi(t)\phi\rangle dt + (v(0), \psi(0)\phi), \end{aligned} \tag{3.127}$$

by comparing with (3.126), one infers that

$$(v(0) - v_0, \phi)\psi(0) = 0.$$

If we choose $\psi$ with $\psi(0) = 1$, then necessarily,

$$(v(0) - v_0, \phi) = 0, \quad \forall \quad \phi \in V.$$

Then since $V$ is dense in $H$, the above holds for any $\phi \in H$. Since $v(0) - v_0 \in H$, one has $(v(0) - v_0, v(0) - v_0)$ and so $v(0) = v_0$.

The final step is to show $v \in C([0,T];H)$. Let us first recall the following weak continuity result from Temam [105].

Observe from the ODE

$$\frac{d^+}{dt}v(t) + (\nu A + \mathbf{C})v(t) = \hat{f} - B(v(t) + z_\alpha(t)) + \alpha z_\alpha(t) \tag{3.128}$$

and lemma 3.1.7, since each term of the right hand side belongs to $L^2(0, T; V')$ and so $\frac{d^+}{dt}v(t)$ also belongs to $L^2(0, T; V')$, hence it follows from Lemma 3.1.5 that $u$ is a.e. a function continuous from $[0, T]$ into $H$. Thus

$$v \in C([0, T]; H). \tag{3.129}$$

Combine with the earlier result (3.112) we conclude that $v \in L^2(0, T; V) \cap C([0, T]; H)$. Note, the solution is in $L^4(0, T; \mathbb{L}^4(\mathbb{S}^2))$ as well. To see this,

$$\int_0^T |v(t)|^4_{\mathbb{L}^4(\mathbb{S}^2)}$$

$$\leq C \int_0^T |v|^2_{\mathbb{L}^2(\mathbb{S}^2)}|v|^2_V < \infty,$$

due to the interpolation inequality in p.12 [66].

So, the proof of existence of global weak solutions is completed. To complete the proof of Theorem 3.2.11 we now prove uniqueness using the classical argument of Lion and Prodi [74] .

#### 3.3.2.3 Uniqueness of solutions

Suppose $v_1$, $v_2$ are two solutions of (3.75) with the same initial condition. Let $w = v_1 - v_2$, then $w$ satisfies

$$\begin{cases} \partial_t w + \nu A w = -B(w, z) - B(z, w) - B(w, v_1) - B(v_2, z) - \mathbf{C}w, \\ w(0) = 0. \end{cases} \tag{3.130}$$

Multiply (3.130) both sides with $w$ and integrate against $w$, using Lemma 3.1.5, equations (3.48) and (3.38), we get

$$\partial_t |w|^2 + 2\nu |w|^2_V = -2b(w, z, w) - 2b(w, v_n, w),$$

Since $|b(w, w, z)| \leq C|w||w|_V|z|_V$ and $|b(w, w, v)| \leq C|w||w|_V|v|_V$, the right hand side

$$\leq C|w||w|_V(|z|_V + |v_1|_V).$$

Then via usual Young inequality with $a = \sqrt{\nu}|w|_V$ and $b = \frac{C}{\sqrt{\nu}}|w|(|z|_V + |v_n|_V)$, one has

$$|b(w, w, v)| \leq \frac{\nu |w|_V}{2} + \frac{C}{2\nu}|w|^2(|z|^2_V + |v_1|^2_V) \tag{3.131}$$

Therefore, by Gronwall lemma one obtains

$$\partial_t |w|^2 \leq \frac{C}{2\nu}(|z|_V^2 + |v_1|_V^2)|w|^2,$$

and combine with $w_0 = v_{1,0} - v_{2,0} = 0$, it is easy to show

$$|w(t)|^2 \leq |w(0)|^2 \exp\left(\frac{C}{2\nu}(\int_0^T |z(t)|_V^2 + |v_1(t)|_V^2)|w(t)|^2 dt\right) < \infty,$$

as $\int_0^T |z(t)|_V^2 + |v_1(t)|^2 dt < \infty$. Now, since $w(0) = 0$, necessarily $w(t)$ must be 0. Therefore Theorem 3.2.11 is proved.

#### 3.3.2.4 Continuous dependence on initial data, noise and force

This subsubsection is devoted to the proof of Theorem 3.2.12. Namely,

**Theorem 4.2.12.** *Assume that,*

$$u_n^0 \to u^0 \quad in \quad H,$$

*and for some $T > 0$,*

$$z_n \to z \quad in \quad L^4([0,T]; \mathbb{L}^4(\mathbb{S}^2) \cap H), \qquad f_n \to f \quad in \quad L^2(0,T;V'). \tag{3.132}$$

*Suppose $v(t,z)u_0$ and $v(t,z_n)u_n^0$ be two solutions of (3.75). Then,*

$$v(\cdot, z_n)u_n^0 \to v(\cdot, z)u_0 \quad in \quad C([0,T];H) \cap L^2(0,T;V).$$

*In particular, $v(T,z_n)u_n^0 \to v(T,z_n)u_0$ in $H$.*

**Proof.** Write

$$v_n(t) = v(t,z_n), v(t) = v(t,z), y_n(t) = v(t,z_n) - v(t,z), \quad t \in [0,T],$$

$$\hat{z}_n = z_n - z, \quad \hat{f}_n = f_n - z.$$

Then it is clear that $y_n$ solves

$$\begin{cases} \partial_t y_n(t) = -\nu A y_n(t) - B(v_n(t) + z_n(t)) + B(v(t) + z(t)) - \mathbf{C} y_n + \alpha \hat{z}_n + \hat{f}_n, \\ y_n(0) = u_n^0 - u_0. \end{cases} \tag{3.133}$$

Since $y_n \in L^2(0,T;V)$ and $\partial_t y_n \in L^2(0,T;V')$, it follows from lemma 3.1.5 that the function $|y_n|^2$ is absolutely continuous on $(0,T)$ and $\frac{1}{2}\partial_t |y_n(t)|^2 = \langle \partial_t y_n(t), y_n(t) \rangle$ holds in the weak sense. Moreover, by equation (3.31) we have $\langle A y_n(t), y_n(t) \rangle = |\nabla y_n(t)|^2$ a.e. on $(0,T)$ and $(\mathbf{C} y_n, y_n) = 0$

and so we arrive with,

$$\frac{1}{2}\partial_t|y_n(t)|^2 + \nu|\nabla y_n(t)|^2 = -b(y_n, v_n, y_n) - b(v, y_n, y_n) - b(\hat{z}_n, v_n, y_n) - b(z, y_n, y_n) - b(v_n, \hat{z}_n, y_n)$$
$$- b(y_n, z, y_n) - b(z_n, \hat{z}_n, y_n) - b(\hat{z}_n, z, y_n) + \alpha(\hat{z}_n, y_n) + (\hat{f}_n, y_n), \quad t \geq 0.$$

Using the Young inequality, we have

$$\begin{aligned} b(y_n, v_n, y_n) &\leq |y_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}|v_n|_V \quad \text{via inequality (3.52)} \\ &\leq |y_n||y_n|_V|v_n|_V \quad \text{via} \quad 3.56 \end{aligned}$$

With $ab = \sqrt{\frac{\nu}{10}}|y_n|_V\sqrt{\frac{10}{\nu}}|v_n|_V|y_n|, p = 2$

$$\leq \frac{\nu}{20}|y_n|^2_V + \frac{5}{\nu}|\mathbf{v}_n|^2_V|y_n|^2.$$

Similarly,

$$\begin{aligned} b(v, y_n, y_n) &\leq |v|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|_V|y_n|_{\mathbb{L}^4(\mathbb{S}^2)} \\ &\leq |v|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|^{3/2}_V|y_n|^{1/2} \end{aligned}$$

Now using Young inequality with $p = 4/3$ and $a = (\frac{15}{\nu})^{-3/4}|y_n|^{3/2}_V$ and $b = (\frac{15}{\nu})^{3/4}|y_n|^{1/2}|v|_{\mathbb{L}^4(\mathbb{S}^2)}$,

$$\leq \frac{\nu}{20}|y_n|^2_V + \frac{15^3}{4\nu^3}|y_n|^2|v|^4_{\mathbb{L}^4(\mathbb{S}^2)},$$

$$\begin{aligned} b(\hat{z}_n, v_n, y_n) &\leq |\hat{z}_n|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|_V|v_n|_{\mathbb{L}^4(\mathbb{S}^2)} \\ &\leq \frac{\nu}{20}|y_n|^2_V + \frac{5}{\nu}|\hat{z}_n|_{\mathbb{L}^4(\mathbb{S}^2)}|v_n||v_n|^2_V, \end{aligned}$$

$$\begin{aligned} b(z, y_n, y_n) &\leq |z|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|_V|y_n|_{\mathbb{L}^4(\mathbb{S}^2)} \\ &\leq |z|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|^{3/2}_V|y_n|^{1/2} \end{aligned}$$

Now using Young inequality with $p = 4/3$ and $a = (\frac{15}{\nu})^{-3/4}|y_n|^{3/2}_V$ and $b = (\frac{15}{\nu})^{3/4}|y_n|^{1/2}|z|_{\mathbb{L}^4(\mathbb{S}^2)}$,

$$\leq \frac{\nu}{20}|y_n|^2_V + \frac{15^3}{4\nu^3}|y_n|^2|z|^4_{\mathbb{L}^4(\mathbb{S}^2)},$$

$$\begin{aligned} b(v_n, \hat{z}_n, y_n) &\leq |\hat{v}_n|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|_V|\hat{z}_n|_{\mathbb{L}^4(\mathbb{S}^2)} \\ &\leq \frac{\nu}{20}|y_n|^2_V + \frac{5}{\nu}|v_n||v_n|^2_V|\hat{z}_n|_{\mathbb{L}^4(\mathbb{S}^2)}, \end{aligned}$$

$$b(y_n, z, y_n) \le |y_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}|z|_V \le |y_n||y_n|_V|\hat{z}_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}$$
$$\le \frac{\nu}{20}|y_n|^2_V + \frac{5}{\nu}|z|^2_V|y_n|^2,$$

$$b(z_n, \hat{z}_n, y_n) \le |z_n|_{\mathbb{L}^4(\mathbb{S}^2)}|y_n|_V|\hat{z}_n|_{\mathbb{L}^4(\mathbb{S}^2)}$$
$$\le \frac{\nu}{20}|y_n|^2_V + \frac{5}{\nu}|z_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n|^2_{\mathbb{L}^4(\mathbb{S}^2)},$$

$$\alpha(\hat{z}_n, y_n) \le \alpha|y_n|_V|\hat{z}_n|_{V'}$$
$$\le \frac{\nu}{20}|y_n|^2_V + \frac{5\alpha^2}{\nu}|\hat{z}_n|^2_{V'},$$

$$(\hat{f}, y_n) \le |y_n|_V|f_n|_{V'}$$
$$\le \frac{\nu}{20}|y_n|^2_V + \frac{5}{\nu}|\hat{f}_n|^2_{V'}.$$

Hence we have,

$$\begin{aligned}\partial_t|y_n|^2 + \nu|y_n|^2_V \le{} & \frac{10}{\nu}|v_n|^2_V|y_n|^2 + \frac{15^3}{2\nu^3}|y_n|^2|v|^4_{\mathbb{L}^4(\mathbb{S}^2)} \\ & + \frac{10}{\nu}|\hat{z}_n|^4_{\mathbb{L}^4(\mathbb{S}^2)}|v_n||v_n|_V + \frac{15^3}{2\nu^3}|y_n|^2|z|^4_{\mathbb{L}^4(\mathbb{S}^2)} \\ & + \frac{10}{\nu}|v_n|_V|v_n|^2|\hat{z}_n|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{10}{\nu}|z|^2_V|y_n|^2 \\ & \frac{10}{\nu}|\hat{z}_n|^4_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{10}{\nu}|z|^4_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n|^4_{\mathbb{L}^4(\mathbb{S}^2)} \\ & \frac{10\alpha^2}{\nu}|\hat{z}_n|^2_{V'} + \frac{10}{\nu}|\hat{f}_n|^2_{V'}.\end{aligned}$$

Integrate over 0 to $t$, one gets

$$|y_n(t)|^2 + \nu\int_0^t |y_n(s)|^2_V ds \le |y_n(0)|^2 + \frac{10}{\nu}\int_0^t \beta_n(s)ds + \int_0^t \gamma_n(s)|y_n(s)|^2 ds, \quad t \in [0, T], \quad (3.134)$$

where

$$\begin{aligned}\beta_n ={} & |\hat{z}_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}|v_n||v_n|_V + |v_n||v_n|_V|\hat{z}_n|^2_{\mathbb{L}^4(\mathbb{S}^2)} + |z_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n|^2_{\mathbb{L}^4(\mathbb{S}^2)} \\ & + |z_n|^2_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n|^2_{\mathbb{L}^4(\mathbb{S}^2)} + \alpha^2|\hat{z}_n|^2_{V'} + |\hat{f}_n|^2_{V'},\end{aligned}$$

$$\gamma_n = \frac{10}{\nu}|v_n|_V^2 + \frac{15^3}{2\nu^3}|v|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{15^3}{2\nu^3}|z|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{10}{\nu}|z|_V^2.$$

Then Gronwall yields

$$|y_n(t)|^2 \le \left(|y(0)|^2 + \frac{10}{\nu}\int_0^t \beta_n(s)ds\right)\exp\left(\int_0^t \gamma_n(s)ds\right).$$

Note that

$$\begin{aligned}\int_0^T \beta_n(s)ds = &\int_0^T [|\hat{z}_n(s)|^2_{\mathbb{L}^4(\mathbb{S}^2)}|v_n(s)||v_n(s)|_V + |v_n(s)||v_n(s)|_V]\hat{z}_n(s)|^2_{\mathbb{L}^4(\mathbb{S}^2)} \\ &+ |z_n(s)|^2_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n(s)|^2_{\mathbb{L}^4(\mathbb{S}^2)} + |z_n(s)|^2_{\mathbb{L}^4(\mathbb{S}^2)}|\hat{z}_n(s)|^2_{\mathbb{L}^4(\mathbb{S}^2)} + \alpha^2|\hat{z}_n(s)|^2_{V'} + |\hat{f}_n(s)|^2_{V'}]ds \\ &\le [2|v_n|_{L^\infty(0,T;H)}|v_n|_{L^2(0,T;V)} + |z_n|^2_{L^4(0,T;\mathbb{L}^4)} + |z|^2_{L^4(0,T;\mathbb{L}^4)}]|\hat{z}_n|^2_{L^4(0,T;\mathbb{L}^4)} \\ &+ \alpha^2|\hat{z}_n|^2_{L^2(0,T;V')} + |\hat{f}_n|^2_{L^2(0,T;V')}.\end{aligned}$$

Hence, by usual continuity argument , pass the limit through the integral one gets

$$\int_0^T \beta_n(s)ds \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, since $|y_n(0)| \to 0$ as $n \to \infty$ and for some finite constant $C$ one has

$$\begin{aligned}\int_0^T \gamma_n(s)ds &= \int_0^T \left(\frac{10}{\nu}|v_n|_V^2 + \frac{15^3}{2\nu^3}|v|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{15^3}{2\nu^3}|z|^4_{\mathbb{L}^4(\mathbb{S}^2)} + \frac{10}{\nu}|z|_V^2\right)ds \\ &\le C.\end{aligned}$$

Hence $y_n(t) \to 0$ in $H$ as $n \to \infty$ uniformly in $t \in [0, T]$. In other words,

$$v(\cdot, z_n)u_n^0 \to v(\cdot, z)u_0 \quad \text{in} \quad C([0, T]; H).$$

From inequality (3.134), we also have

$$\begin{aligned}\nu\int_0^T |y_n(s)|_V^2 &\le |y_n(0)|^2 + \frac{10}{\nu}\int_0^T \beta_n(s)ds + \int_0^T \gamma_n(s)|y_n(s)|^2 ds \\ &\le |y_n(0)|^2 + \frac{10}{\nu}\int_0^T \beta_n(s)ds + \sup_{s\in[0,T]}|y_n(s)|^2\int_0^T \gamma_n(s)ds.\end{aligned}$$

Therefore,

$$\int_0^T |y_n(s)|_V^2 ds \to 0 \quad \text{as} \quad n \to \infty. \tag{3.135}$$

Hence

$$v(\cdot, z_n)u_n^0 \to v(\cdot, z)u_0 \quad \text{in} \quad L^2([0, T]; V).$$

and Theorem 3.2.12 is proved. □

## 3.4 Proof of Theorem 3.2.15: Strong solutions

Suppose now $f \in H$, in what proceeds we will show that if $u_0 \in V$ then we obtain a more regular kind of solution and deduce that if $v_0 \in H$ then $v(t) \in V$ for every $t > 0$. In this chapter, we will construct a unique global strong solution (in PDE sense).

The proof of Theorem 3.2.15 follows closely to Theorem 3.1 in [19]. However in the proof in [19] there is no Coriolis force and additive noise whereas here there are. In particular our constants in the proof now depend on $|F(t)|$ and $|z(t)|$ and $|z(t)|_V$, but not on the Coriolis term due to the antisymmetric condition $(\mathbf{C}v, Av) = 0$.

**Remark.** One can alternatively prove Theorem 3.2.15 via the usual Galerkin approximation which we used in the proof of weak variational solution.

### 3.4.1 Existence and uniqueness of strong solution with $v_0 \in V$

The following function spaces are introduced for convenience.

**Definition 3.4.1.** The spaces

$$X_T := C(0,T;H) \cap L^2(0,T;V), \tag{3.136}$$

$$Y_T = C(0,T;V) \cap L^2(0,T;D(A)) \tag{3.137}$$

are endowed with the norm

$$|\cdot|_{X_T} := |\cdot|_{C(0,T;H)} + |\cdot|_{L^2(0,T;V)},$$

$$|\cdot|_{Y_T} := |\cdot|_{C(0,T;V)} + |\cdot|_{L^2(0,T;D(A))}.$$

Or explicitly,

$$|f|^2_{X_T} = \sup_{0\le t\le T} |f(t)|^2 + \int_0^T |f(s)|^2_V ds,$$

$$|f|^2_{Y_T} = \sup_{0\le t\le T} |f(t)|^2_V + \int_0^T |Af(s)|^2 ds.$$

Let $\mathcal{K}$ be the map in $Y_T$ defined by

$$\mathcal{K}(u)(t) = \int_0^t S(t-s)B(u(s))ds, \quad t \in [0,T],\ u \in Y_T.$$

The following is a crucial lemma for the proof of existence and uniqueness.

**Lemma 3.4.2.** *There exists $c > 0$ such that for every $u, v \in Y_T$,*

$$|\mathcal{K}(u)|^2_{Y_T} \le c|u|^2_{Y_T}\sqrt{T},$$

$$|\mathcal{K}(u) - \mathcal{K}(v)|^2_{Y_T} \le c|u-v|^2_{Y_T}(|u|^2_{Y_T} + |v|^2_{Y_T})\sqrt{T}.$$

**Proof.** Recall classical facts due to Lions [73],

- for any $f \in L^2(0,T;H)$, the function $t \mapsto x(t) = \int_0^t S(t-s)f(s)ds$ belongs to $Y_T$ and
- the map $f \mapsto x$ is continuous from $L^2(0,T;H)$ to $Y_T$.

We remark that the second fact implies $\int_0^t |f(s)|^2_H ds < \infty$ Now because $B(u) \in L^2(0,T;H)$, that is $\int_0^t |B(u(s))|^2_H ds$, using the previous classical facts, combine with (3.55) one has,

$$\begin{aligned}
|\mathcal{K}(u)|^2_{Y_T} &\le c_1 \int_0^T |B(u(s))|^2_H ds \\
&\le c_2 \int_0^T |u|^2_V |u|_V |Au| dt \\
&\le c_2 \sup_{0\le t\le T} |u(t)|^2_V \int_0^T |u(t)|_V |Au(t)| dt \\
&\le \frac{c_2}{2} \sup_{0\le t\le T} |u(t)|^2_V \left( \int_0^T |u(t)|^2_V + |Au(t)|^2 dt \right) \\
&\le c_3 |u|^4_{Y_T} \sqrt{T}.
\end{aligned}$$

Similarly, combine Lions' results and (3.55), one has

$$\begin{aligned}
|\mathcal{K}(u) - \mathcal{K}(v)|^2_{Y_T} &\le c_4 \int_0^T |B(u-v,u) + B(v,u-v)|^2_H dt \\
&\le c_5 \int_0^T |B(u-v,u)|^2_H + |B(v,u-v)|^2_H dt \\
&\le c_5 \int_0^T c_7 |u-v|^2_V |u|_V |Au| + c_8 |u-v|^2_V |v|_V |Av| dt \\
&\le c|u-v|^2_{Y_T}(|u|^2_{Y_T} + |v|^2_{Y_T})\sqrt{T}.
\end{aligned}$$

□

**Lemma 3.4.3.** *Assume that $\alpha \ge 0$, $z \in L^4_{loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in H$ and $v_0 \in V$. Then, there exists unique solution of (3.73) in the space $C(0,T;V) \cap L^2(0,T;D(A))$. for all $T > 0$.*

**Proof.** First let us prove local existence and uniqueness. Let $Y_\tau = C(0,\tau;V)\cap L^2(0,\tau;D(A))$ be equipped with the norm

$$|f|^2_{Y_\tau} = \sup_{t\le\tau}|f(t)|^2 + \int_0^\tau |Af(s)|^2\,ds,$$

and Let $\Gamma$ be a nonlinear mapping in $Y_\tau$ as

$$(\Gamma v)(t) = S(t)v_0 + \int_0^t S(t-s)(f - B(v(s)+z(s)) + \alpha z(s))ds.$$

Now recall the following classical result due to Lion.

$$\begin{cases} A1 & S(\cdot)v_0 \in Y_\tau,\ \forall\ v_0 \in H, \tau > 0; \\ A2 & \text{The map } t \mapsto x(t) = \int_0^t S(t-s)f(s)ds \text{ belongs to } Y_\tau \text{ for all } L^2(0,\tau;H); \\ A3 & \text{The mapping } f \mapsto x \text{ is continuous from } L^2(0,\tau;H) \text{ to } Y_\tau. \end{cases}$$

Note, our assumption $z(t) \in L^4([0,\infty);L^4(\mathbb{S}^2)\cap H)$ implies $z(t)\in Y_\tau$ as $z(t)$ is square integrable and $V$ can be continuously embedded into $L^4(\mathbb{S}^2)$ .
The first step is to show $\Gamma$ is well defined. Using assumptions A1 and A2 and the assumption for $z(t)$, together with Young inequality, one can show

$$|\Gamma|^2_{Y_\tau} \le c\,|S(t)v_0|^2_{Y_\tau} + c\left|\int_0^t S(t-s)B(v(s)+z(s))ds\right|^2_{Y_\tau} + c\left|\int_0^t S(t-s)f ds\right|^2_{Y_\tau} + c\alpha\left|\int_0^t S(t-s)z(s)\right|^2_{Y_\tau}.$$

For some different constant $c$. Now due to $A_1$ and $A_2$, the first and third terms are finite, due to $A_2$ and the trilinear inequality (3.52), the second term is finite, and the last term also finite due to the assumption on $z(t)$

$$|\Gamma|^2_{Y_\tau} \le c_1 + c_2|v|^4_{Y_\tau} + c_3 + c_4. \tag{3.138}$$

Whence the map $\Gamma v$ is well defined in $Y_\tau$ and $\Gamma$ maps $Y_\tau$ into $Y_\tau$ itself.
Now we have

$$\begin{aligned} &|\Gamma(v_1) - \Gamma(v_2)|^2_{Y_\tau} \\ &\le |\int_0^\tau S(t-s)(B(v_1(s)+z(s)) - B(v_2(s)+z))ds|^2_{Y_\tau} \\ &\le c_6|v_1-v_2|^2_{Y_\tau}(|v_1+z|^2_{Y_\tau} + |v_2+z|^2_{Y_\tau})\sqrt{\tau}, \end{aligned}$$

for all $v_1$, $v_2$ and $z$ in $Y_\tau$. Therefore, for sufficiently small $\tau > 0$, $\Gamma$ is a contraction in a closed ball of $Y_\tau$, yielding existence and uniqueness of a local solution of (3.76) in $Y_\tau$. That is, the solutions are bounded in $V$ on some short time $[0,\tau)$.

**Remark.** If the following map

$$(\Gamma u)(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)f ds + \int_0^t S(t-s)GdL(s)$$

is used to prove contraction. Then one would have to assume

$$\int_0^T |Az(t)|^2 dt < \infty.$$

The local existence and uniqueness results indicates that the solution can be extended up to the maximal lifetime $T_{f,z}$ and then is well defined on the right open interval $[0, T_{f,z})$. Next, we will prove the local solution may be continued to the global solution valid for all $t > 0$, in the class of weak solutions satisfying a certain energy inequality. (This is an analogue of the well-known fact that the deterministic 2D NSE has a unique global strong solution. See for instance Theorem 7.4 Foias and Temam [55])
It suffices to find an uniform a priori estimate for the solution $v$ in the space $Y_{T_0}$ such that for any $T_0 \in [0, T_{f,z})$:

$$|v|^2_{Y_{T_0}} \le C \quad \text{for all} \quad T_0 \in [0, T_{f,z}), \tag{3.139}$$

where $C$ is independent on $T_0$. This uniform a priori estimate along with the local existence uniqueness proved earlier yields the unique global solution $u$ in $Y_{T,z}$ indeed exist globally in time. Hence one can deduce that the solution is well defined up to time $t = T_{f,z}$, at this point in time the iterated process could be repeated and the solution can be found in $[T_{f,z}, 2T_{f,z}]$ and so forth, hence in $C(0,\infty; V) \cap L^2_{\text{loc}}(0, \infty; D(A))$. To prove (3.139), we first need to show

$$|v|_{X_{T_0}} \le c_0.$$

Toward the above end, we work with a modified version of (3.75)

$$\begin{cases} \partial_t v + \nu A v = -B(v) - B(v,z) - B(z,v) - \mathbf{C}v + F, \\ v(0) = v_0, \end{cases} \tag{3.140}$$

where $F = -B(z) + \alpha z + f$ is an element of $H$ since the $H$ norm of all its three terms is bounded. Now multiply both sides with $v$, integrate over $\mathbb{S}^2$, one gets

$$\begin{aligned} \partial_t |v|^2 + \nu |v|^2_V &= -b(v,v,v) - b(v,z,v) - b(z,v,v) - (\mathbf{C}v, v) + \langle F, v\rangle \\ &= b(v,v,z) + (F, v). \end{aligned}$$

Now by (3.50), one has

$$|b(v,v,z)| \le c|v||v|_V|z|$$

then apply Young inequality with $ab = \sqrt{\frac{\epsilon}{2}}|v|_V|v|\sqrt{\frac{2}{\epsilon}}|z|_V$ it follows that

$$\leq \frac{\epsilon|v|_V^2}{4} + \frac{1}{\epsilon}|v|^2|z|_V^2.$$

On the other hand,

$$(F(t), v) = |F(t)||v| \leq \frac{1}{\epsilon}|F(t)|^2 + \frac{\epsilon}{4}|v|^2.$$

So

$$\partial_t|v|^2 + (2\nu - \frac{\epsilon}{2})|v|_V^2 \leq \frac{2}{\epsilon}|v|^2|z|_V^2 + \frac{2}{\epsilon}|F(t)|^2 + \frac{\epsilon}{2}|v|^2 \tag{3.141}$$

for all $\epsilon > 0$.

By integrating in $t$ from 0 to $T$, after dropping out unnecessary terms,

$$\int_0^T |v(t)|_V^2 \leq \frac{1}{2\nu - \frac{\epsilon}{2}}\left(|v(0)|^2 + \frac{2}{\epsilon}\int_0^T |v(t)|^2|z(t)|_V^2 dt + \frac{2}{\epsilon}\int_0^T |F(t)|^2 dt + \frac{\epsilon}{2}\int_0^T |v(t)|^2 dt\right) \leq K_1, \tag{3.142}$$

since $v(0) = u_0$

$$K_1 = K_1(u_0, F, \nu, T, z).$$

On the other hand, by integrating in $t$ of (3.141) from 0 to $s$, $0 < s < T$, we obtain

$$|v(s)|^2 \leq |u_0|^2 + \frac{2}{\epsilon}\int_0^s |v(t)|^2|z(t)|_V^2 dt + \frac{2}{\epsilon}\int_0^s |F(t)|^2 dt + \frac{\epsilon}{2}\int_0^s |v(t)|^2 dt,$$

$$\sup_{s\in[0,T_{f,z}]} |v(s)|^2 \leq K_2,$$

$$K_2 = K_2(u_0, F, \nu, T, z) = (2\nu - \frac{\epsilon}{2})K_1.$$

Hence, for any $\epsilon$ such that $\frac{\epsilon}{2} < 2\nu$, apply Gronwall lemma to

$$\partial_t|v|^2 \leq \left(\frac{2}{\epsilon}|z|_V^2 + \frac{\epsilon}{2}\right)|v|^2 + \frac{2}{\epsilon}|F(t)|^2,$$

one obtains

$$|v(t)|^2 \leq |v(0)|^2 \exp\left(\int_0^t \frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}d\tau\right)|v|^2 + \int_0^t \frac{2}{\epsilon}|F(s)|^2 \exp(\int_s^t \left(\frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}\right)d\tau)ds,$$

and so

$$\sup_{t\in[0,T_{f,z}]} |v(t)|^2 \leq |v(0)|^2 \exp\left(\int_0^{T_{f,z}} \frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}d\tau\right) + \int_0^{T_{f,z}} \frac{2}{\epsilon}|F(s)|^2 \exp(\int_s^{T_{f,z}} \left(\frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}\right)d\tau)ds.$$

To avoid clumsiness, write momentarily $T_{f,z} = T$. Let

$$\psi_T(z) = \exp\left(\int_0^T \frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}d\tau\right) < \infty, \quad c_F = \int_0^T \frac{2}{\epsilon}|F(s)|^2 \exp\left(\int_s^T \left(\frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}\right) d\tau\right) ds. \tag{3.143}$$

So

$$\sup_{t\in[0,T]} |v(t)|^2 \leq |v(0)|^2\psi_T(z) + c_F, \tag{3.144}$$

which implies

$$v \in L^\infty([0,T];H). \tag{3.145}$$

Now integrate

$$\partial_t|v|^2 + \nu|v|_V^2 \leq \left(\frac{2}{\epsilon}|z|_V^2 + \frac{\epsilon}{2}\right)|v|^2 + \frac{2}{\epsilon}|F(t)|^2, \tag{3.146}$$

from 0 to $T$ one gets

$$|v(T)|^2 + \nu\int_0^T |v(t)|_V^2 dt \leq \left(\psi_T(z)|v(0)|^2 + c_F\right)\int_0^T \left(\frac{2}{\epsilon}|z(t)|^2 + \frac{\epsilon}{2}\right) dt + \frac{2}{\epsilon}\int_0^T |F(t)|^2 dt + |v(0)|^2, \tag{3.147}$$

which implies

$$v \in L^2([0,T];V), \tag{3.148}$$

and $v$ is indeed a weak solution. To show $v \in C([0,T];H)$, note that $A: V \to V'$ is bounded and $Av \in L^2([0,T];V')$. Then $F \in L^2([0,T];V')$ since $z \in L^4([0,T];L^4(\mathbb{S}^2)\cap H)$ which can be continuously embedded into $V'$, and the terms $B(z)$, $B(v,z)$, $B(z,v)$ all in $L^2([0,T];V')$. Combine these facts and (3.148), invoke lemma 4.1 in [14] we conclude that $v \in C([0,T];H)$.
The uniform apriori estimate (3.147) implies that the solution is well defined up to time $t = T_{f,z}$. The iterative process may be repeated start from $t = T_{f,z}$ with the initial condition $z(t)$ and the solution is uniquenely extended to $[0, 2T_{f,z}]$ and so on to arbitrary large time.
Now, multiply (3.140) both sides with $Av$, noting again the classical fact $\frac{1}{2}\partial_t|v(t)|^2 = (\partial_t v(t), v(t))$ and $(Cv, Av) = 0$, integrate over $\mathbb{S}^2$, one gets

$$(\partial_t v, Av) + \nu(Av, Av) = -b(v,v,Av) - b(v,z,Av) - b(z,v,Av) + \langle F(t), Av(t)\rangle$$

$$\Longrightarrow \frac{1}{2}\frac{d^+}{dt}|v|^2 + \nu|Av|^2 = -b(v(t),v(t),Av(t)) - b(v(t),z(t),Av(t)) - b(z(t),v(t),Av(t)) + \langle F(t), Av(t)\rangle. \tag{3.149}$$

Now,

$$|b(v,v,Av)| \leq C|v|^{\frac{1}{2}}|v|_V|Av|^{\frac{3}{2}} \quad \forall\, v \in V, v \in D(A),$$

$$|b(v,z,Av)| \leq C|v|^{\frac{1}{2}}|v|_V^{\frac{1}{2}}|z|_V^{\frac{1}{2}}|Av|^{\frac{3}{2}} \quad \forall\, v \in V, v \in D(A),$$

$$|b(z,v,Av)| \leq C|z|^{\frac{1}{2}}|z|^{\frac{1}{2}}|v|_V^{\frac{1}{2}}|Av|^{\frac{3}{2}} \quad \forall\, z \in V, v \in D(A).$$

Also,

$$(F(t), Av) \leq \frac{\epsilon}{4}|Av(t)|^2 + \frac{1}{\epsilon}|F(t)|^2.$$

Furthermore, using Young inequality with the choice $p = \frac{4}{3}$ and $ab = (\epsilon p)^{\frac{1}{p}}|Av|^{3/2}\epsilon p)^{-\frac{1}{p}}|v|^{1/2}|v|_V$, the above estimates of the three bilinear terms become

$$\begin{aligned} |b(v,v,Av)| &\leq C|v|^{\frac{1}{2}}|v|_V|Av|^{\frac{3}{2}} \\ &\leq \epsilon|Av|^2 + C(\epsilon)|v|^2|v|_V^4, \end{aligned}$$

$$\begin{aligned} |b(v,z,Av)| &\leq C|v|^{\frac{1}{2}}|v|_V^{\frac{1}{2}}|z|_V^{\frac{1}{2}}|Av|^{\frac{3}{2}} \\ &\leq \epsilon|Av|^2 + C(\epsilon)|v|^2|v|_V^2|z|_V^2, \end{aligned}$$

$$\begin{aligned} |b(z,v,Av)| &\leq C|z|^{\frac{1}{2}}|z|_V^{\frac{1}{2}}|v|_V^{\frac{1}{2}}|Av|^{\frac{3}{2}} \\ &\leq \epsilon|Av|^2 + C(\epsilon)|z|^2|z|_V^2|v|_V^2. \end{aligned}$$

Therefore,

$$\frac{d^+}{dt}|v|_V^2 + (2\nu - 3\epsilon - \frac{\epsilon}{4})|Av|^2 \leq C(\varepsilon)(|v|^2|v|_V^4 + |v|^2|v|_V^2|z|_V^2 + |z|^2|z|_V^2|v|_V^2) + \frac{1}{\epsilon}|F(t)|^2. \tag{3.150}$$

Momentarily dropping the term $|Av(t)|^2$, we have the differential inequality,

$$y' \leq a + \theta y,$$

$$y(t) = |v|_V^2, \quad a(t) = \frac{1}{\nu}|F(t)|^2, \theta(t) = C(\epsilon)(|v|^2|v|_V^2 + |v|^2|z|_V^2 + |z|^2|z|_V^2).$$

Then for any $\epsilon$ such that $\epsilon < \frac{8}{13}\nu$, using Gronwall lemma , one has

$$\frac{d^+}{dt}\left(y(t)e^{-\int_0^t \theta(\tau)d\tau}\right) \leq a(t)e^{-\int_0^t \theta(\tau)d\tau}ds$$

$$\begin{aligned} |v(t)|_V^2 \leq |v(0)|_V^2 \exp\left(\int_0^t C(\epsilon)(|v(\tau)|^2|v(\tau)|_V^2 + |v(\tau)|^2|z(\tau)|_V^2 + |z(\tau)|^2|z(\tau)|_V^2)d\tau\right) \\ + \frac{1}{\nu}\int_0^t |F(s)|^2 \exp\left(\int_s^t C(\epsilon)(|v(\tau)|^2|v(\tau)|_V^2 + |v(\tau)|^2|z(\tau)|_V^2 + |z(\tau)|^2|z(\tau)|_V^2)d\tau\right) ds \end{aligned}$$

$$\sup_{t\in[0,T]} |v(t)|_V^2 \leq K_3, \tag{3.151}$$

$$K_3 = K_3(u_0, F, \nu, T, z) = \left( |v(0)|_V^2 + \frac{1}{\nu}\int_0^T |F(s)|^2 ds \right) \exp(C(\epsilon)K_2K_1),$$

which implies

$$v \in L^\infty(0, T; V). \tag{3.152}$$

Let us now come back to (3.150), which we integrate from 0 to $T$, after dropping some unnecessary terms, we have

$$\int_0^T |Av(t)|^2 dt \le K_4,$$

and

$$\begin{aligned} K_4 &= K_4(u_0, F, \nu, z, T) \\ &= \frac{1}{2\nu - 3\epsilon - \frac{\epsilon}{4}}(|u_0|^2 + C(\epsilon) \sup_{t\in[0,T]} |v(t)|^2 |v(t)|_V^4 + C(\epsilon) \sup_{t\in[0,T]} |v(t)|^2 |v(t)|_V^2 |z(t)|_V^2 \\ &+ C(\epsilon) \sup_{t\in[0,T]} |z(t)|^2 |z(t)|_V^2 |v(t)|_V^2 + \frac{1}{\epsilon}\int_0^T |F(t)|^2) dt. \end{aligned}$$

As

$$\begin{aligned} \sup_{t\in[0,T]} |v(t)|^2 &\le K_2, \\ \sup_{t\in[0,T]} |v(t)|_V^4 &\le K_3^2, \\ |z(t)|_V^2 &\le C_1, \\ \sup_{t\in[0,T]} |z(t)|^2 &\le C_2. \end{aligned}$$

So,

$$K_4 = \frac{1}{2\nu - 3\epsilon - \frac{\epsilon}{4}}(|u_0|^2 + C(\epsilon)K_2K_3^2 + C(\epsilon)K_2K_3C_1 + C(\epsilon)C_2C_1K_3 + \frac{1}{\epsilon}\int_0^T |F(t)|^2) dt.$$

This implies

$$v \in L^2(0, T_{f,z}; D(A)). \tag{3.153}$$

It remains to show $v \in C([0,T]; V)$. Note, the fact that the solution with $v_0 \in V$ is in $L^2([0,T]; V)$ implies that a.e. in $[0,T]$, $v(t) \in V$. Moreover, since $v(t) \in C([0,T]; H)$ as previously deduced, and is unique as proved in step 1. It follows that $u \in C([0,T]; V)$

With the uniform a priori estimate along with the local existence uniqueness in step 1, we conclude that there exists unique $u \in C(0,\infty; H) \cap L^2(0,\infty; V) \subset C(0,\infty; V) \cap L^2(0,\infty; D(A))$, for any given $u_0 \in V$, $f \in H$, $z(t) \in L^4_{\text{loc}}([0,\infty);\mathbb{L}^4(\mathbb{S}^2)\cap H)$. Moreover, our promising a priori bound (3.151) yields $T = \infty$. □

### 3.4.2 Existence and uniqueness of strong solution with $v_0 \in H$

**Corollary 2.** *If $f \in H$, $v_0 \in H$, $z(t) \in L^4_{\text{loc}}([0,\infty);\mathbb{L}^4(\mathbb{S}^2)\cap H)$, then $v(t) \in V$ for all $t > 0$.*

We follow the proof in [19]. The idea stems from standard approximation method commonly used in PDE theory. In view of the a priori estimate (3.150) one takes approximated solution to (3.73) in $Y_T$, show the approximates converge. Then show the limit function indeed satisfies (3.73).

Let $(v_{0,n}) \subset V$ be a sequence converging to $v_0$ in $H$. For all $n \in \mathbb{N}$, let $v_n$ be solution of equation (3.73) in $Y_T$ corresponding to intial data $v_{0,n}$. Similar to the case when $v_0 \in V$, one can find constant such that $|v_n|_{X_T} \le c, \quad \forall\, n \in \mathbb{N}$. Follow the same lines as in the proofs of (3.145) and (3.148), $v_n$ can be proved to be a weak solution.

Moreover, for $n, m \in \mathbb{N}$, take $v_{n,m} = v_n - v_m$ with $v^0_{n,m} = v^0_n - v^0_m$. Then $v_{n,m}$ is the solution of

$$\begin{cases} \partial_t v_{n,m} + \nu A v_{n,m} = -B(v_{n,m}, z) - B(z, v_{n,m}) - B(v_{n,m}, v_n) - B(v_m, v_{n,m}) - \mathbf{C} v_{n,m}, \\ v_{n,m}(0) = v^0_n - v^0_m \end{cases} \tag{3.154}$$

Multiply (3.154) both sides with $v_{n,m}$ and integrate against $v_{n,m}$, using again Lemma 3.1.5 and (3.48) and noting (3.38) one gets

$$\partial_t |v_{n,m}|^2 + 2\nu |v_{n,m}|^2_V = -2b(v_{n,m}, z, v_{n,m}) - 2b(v_{n,m}, v_n, v_{n,m})$$

Since $|b(w,w,z)| \le C|w||w|_V|z|_V$ and $|b(w,w,v)| \le C|w||w|_V|v|_V$

$$\le C|v_{n,m}||v_{n,m}|_V(|z|_V + |v_n|_V)$$

Then via usual Young inequality with $a = \epsilon|v_{n,m}|_V$ and $b = \frac{C}{\sqrt{\epsilon}}|v_{n,m}|(|z|_V + |v_n|_V)$,

$$\le \frac{\epsilon|v_{n,m}|_V}{2} + \frac{C}{2\epsilon}|v_{n,m}|^2(|z|^2_V + |v_n|^2_V). \tag{3.155}$$

Therefore, for any $\epsilon > 0$ such that $\frac{\epsilon}{2} < 2\nu$, one applies Gronwall lemma to

$$\partial_t |v_{n,m}|^2 \le \frac{C}{2\epsilon}(|z|^2_V + |v_n|^2_V)|v_{n,m}|^2,$$

and combine with $v^0_{n,m} = v^0_n - v^0_m$, it is easy to show

$$|v_{n,m}(t)|^2 \le |v_{n,m}(0)|^2 \exp\left(\frac{C}{2\epsilon}(\int_0^T |z(t)|^2_V + |v_n(t)|^2_V)|v_{n,m}(t)|^2 dt\right) < \infty,$$

as $\int_0^T |z(t)|^2_V + |v_n(t)|^2_V < \infty$. Hence $v_{n,m}$ converges in $T$ and so is Cauchy in $T$. That is, for any $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|v_n - v_m| < \epsilon$ whenever $n, m \ge N$.

Let the limit of of $v_n$ be $v$. It remains to show $v$ indeed satisfies (3.73).

Let $v_n$ be the solution to

$$v_n(t) = S(t)v_{0,n} - \int_0^t S(t-s)(B(u_n(s)))ds + \alpha \int_0^t z_n(s)ds, \tag{3.156}$$

where $z_n = \int_0^t S(t-s)GdL_n(t)$. We would like to show

$$\lim_{n\to\infty} u_n(t) = S(t)u_0 - \int_0^t S(t-s)(B(u(s)))ds + \int_0^t S(t-s)fds + \alpha \int_0^t z(s)ds. \tag{3.157}$$

Assume $f_n \to f$ in $L^2(0,T;H)$ , $z_n = \int_0^t S(t-s)GdL_n(t) \to z$ in $L^4([0,T];L^4(\mathbb{S}^2)\cap H)$, we would like to check if

$$\lim_{n\to\infty} B(u_n) = B(u) \quad \text{in} \quad H. \tag{3.158}$$

For this, note first that

$$\begin{aligned}
\left||u_n|_V^2 - |u|_V^2\right| &= |(u_n,u_n) - (u,u)| \\
&= |(u_n,u_n)_V - (u,u_n)_V + (u,u_n)_V - (u,u)_V| \\
&= |(u_n,u_n)_V - (u,u_n)_V| + |(u,u_n)_V - (u,u)_V| \\
&\leq |u_n - u|_V|u_n|_V + |u|_V|u_n - u|_V.
\end{aligned}$$

Now $|u_n|_V$ is Cauchy and so is bounded. So $u_n$ converges to $u$ in $V$ as $n \to \infty$. Then using (3.50) one deduces that

$$\begin{aligned}
&|B(u_n) - B(u)| \\
&= |B(u_n,u_n) - B(u_n,u) + B(u_n,u) - B(u,u)| \leq C(|u_n|_V^2 + |u_n|_V^2|u| + |u|_V^2) \to C|u|_V^2.
\end{aligned}$$

Now similar to the earlier work on proving contraction we have,

$$\begin{aligned}
&|B(u_n(s)) - B(u(s))|_{Y_T}^2 \\
&\leq \left|\int_0^t S(t-s)(B(u_n(s)) - B(u(s)))ds\right|_{Y_T}^2 \\
&\leq c\int_0^T |B(u_n(s)) - B(u(s))|^2 ds \\
&\leq c|u|_T^2\sqrt{T}.
\end{aligned}$$

Therefore, $B(u_n) - B(u)$ is in $L^2(0,T;H)$. Now by continuity argument again, one has

$$\lim_{n\to\infty}\int_0^T S(t-s)B(u_n(s))ds = \int_0^T S(t-s)B(u(s))ds,$$

and

$$\lim_{n\to\infty}\int_0^T S(t-s)f_n(s)ds = \int_0^T S(t-s)f(s)ds.$$

Combine with the assumptions

$$\lim_{n\to\infty} S(t)u_{0,n} = S(t)u_0,$$
$$\lim_{n\to\infty} z_n(t) = z(t),$$

one deduces that

$$\lim_{n\to\infty} v_n(t) = v(t).$$

and there exists a solution to (3.73). However, the solution constructed as the limits of $u_n$ leaves open the possibility that there is more than one limit. So we will now prove $u$ is unique. The idea is analogous to proving (3.155). Nevertheless we detail as following. Suppose $v_1$, $v_2$ are two solutions of (3.75) with the same initial condition. Let $w = v_1 - v_2$, then $w$ satisfies

$$\begin{cases} \partial_t w + \nu A w = -B(w,z) - B(z,w) - B(w,v_1) - B(v_2,w), \\ w(0) = 0. \end{cases} \tag{3.159}$$

Multiply (3.159) both sides with $w$ and integrate against $w$, using the identities $\partial_t|v(t)|^2 = 2\langle \partial_t v(t), v(t)\rangle$ again in Temam and (3.48) one gets

$$\partial_t|w|^2 + 2\nu|w|_V^2 = -2b(w,z,w) - 2b(w,v_1,w)$$

Since $|b(w,w,z)| \leq C|w||w|_V|z|_V$ and $|b(w,w,v)| \leq C|w||w|_V|v|_V$

$$\leq C|w||w|_V(|z|_V + |v_1|_V)$$

Then via usual Young inequality with $a = \sqrt{\epsilon}|w|_V$ and $b = \frac{C}{\sqrt{\epsilon}}|w|(|z|_V + |v_n|_V)$

$$\leq \frac{\epsilon|w|_V}{2} + \frac{C}{2\epsilon}|w|^2(|z|_V^2 + |v_1|_V^2). \tag{3.160}$$

Therefore, for any $\epsilon > 0$ such that $\frac{\epsilon}{2} < 2\nu$, one applies Gronwall lemma to

$$\partial_t|w|^2 \leq \frac{C}{2\epsilon}(|z|_V^2 + |v_1|_V^2)|w|^2,$$

and combine with $w_0 = v_{1,0} - v_{2,0} = 0$, it follows from Gronwall inequality that

$$|w(t)|^2 \leq |w(0)|^2 \exp\left(\frac{C}{2\epsilon}(\int_0^T |z(t)|_V^2 + |v_1(t)|_V^2)|w(t)|^2 dt\right) < \infty$$

as $\int_0^T |z(t)|_V^2 + |v_1(t)|^2 dt < \infty$. Now, since $w(0) = 0$, necessarily $w(t)$ must be 0.
It remains to show $v \in C((0,T;V)$, as observe from the above energy inequality (3.155), the solution starts with with an initial condition $v_0 \in H$ belongs to $L^2(0,T;V)$. This implies that almost everywhere in $(0,T]$, there must exist a time point $\epsilon$ (and $\epsilon < T$) such that $u(\epsilon) \in V$. Then one may repeat step 2 to another interval $[\epsilon, 2\epsilon], [2\epsilon, 3\epsilon] \cdots$ . and over the whole $[\epsilon, \infty]$. Finally we obtain that $u \in C([\epsilon,T];V) \cap L^2([\epsilon,T];D(A))$ for all $\epsilon > 0$. Note that $T = \infty$ as implied

from the a apriori estimate.

In summary, in this section, we have proved

**Lemma 3.4.4.** *Assume that* $\alpha \geq 0$*,* $z \in L^4_{\text{loc}}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$*,* $f \in H$ *and* $v_0 \in H$*. Then, there exists unique solution of (3.76) in the space* $C(0,T;H) \cap L^2(0,T;V)$*. which belongs to* $C(\epsilon,T;V) \cap L^2_{\text{loc}}(\epsilon,T;D(A))$ *for all* $\epsilon > 0$*. and* $T > 0$*.*

Combine Lemma 3.4.4 with 3.4.3, we have proved theorem 3.2.15.

**Remark.** Continuous dependence on $v_0$, $z$ and $f$ is implied from the point where local existence and uniqueness is attained and hence holds also for global solutions.
Finally, we give an intuitive meaning of Theorem 3.2.15.

**Remark.** The proof of Theorem 3.2.15 shows that the solution $v$, starting from $v_0 \in H$, belongs to $V$ for a.e. $t \geq t_0$; If we take any $\bar{t} \geq t_0$ such that $v(\bar{t}) \in V$, the solution is extended over the interval $[t_0, t_0+\epsilon]$ and is found to be in $D(A)$ as well. One may repeat this step over another intevral $[t_0+\epsilon, t_0+2\epsilon]$, $[t_0+2\epsilon, t_0+3\epsilon]\cdots$. Thus, we obtain that $v \in C([t_0+\epsilon,\infty);V) \cap L^2_{\text{loc}}(t_0+\epsilon, D(A))$.
Furthermore, provided the noise does not degenerate, base on the condition given in the following, we obtained the existence and uniqueness results for the solution to the original equation (3.61). We detail this in the next subsubsection.
If

$$\sum_l \lambda^{\frac{\beta}{2}} |\sigma_l|^\beta < \infty, \tag{3.161}$$

then by Lemma 3.2.4 the process $z$ has version which has left limits and is right continuous in $V$. Recall that $u_t := v_t + z_t$ and for each $T > 0$, define

$$Z_T(\omega) := \sup_{0 \leq t \leq T} |z_t(\omega)|_V, \quad \omega \in \Omega. \tag{3.162}$$

If (3.161) holds then by Lemma 3.2.2 we have

$$\mathbb{E} Z_T < \infty,$$

hence there exists a measurable set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and

$$Z_T(\omega) < \infty, \quad \omega \in \Omega_0\,.$$

Finally, let us study (3.61) for $\omega \in \Omega_0$. Since $z(\cdot,\omega) \in D([0,\infty);V)$, it is of course $z(\cdot,\omega) \in D([0,\infty);H)$. Therefore, by Theorem (3.2.12), $u(\cdot,\omega) = v(\cdot,\omega) + z \cdot (\omega)$ is the unique cádlág solution to (3.61). So, we extend the existence theorem of strong solution for $u$. Moreover, for $\omega \in \Omega_0$ we find that $u(\cdot,\omega) = v(\cdot,\omega) + z(\cdot,\omega)$ is the unique solution to (3.61) in $D([0,\infty);H) \cap D([0,\infty);V)$ which belongs to $D([h,\infty);V) \cap L^2_{\text{loc}}(h,\infty);D(A))$ for all $h > 0$. If $u_0 \in V$, then $u \in D([h,\infty);V) \cap L^2_{\text{loc}}([h,\infty);D(A))$ for all $h > 0$, $T > 0$.
This completes the proof of Theorem 3.2.16.
Since the solution is constructed using Banach Fixed Point Theorem, the continuous dependence on initial data is implied from the existence-uniqueness proof of strong solution in above line. Moreover, our existence uniqueness results work naturally when initial time $t_0 \in \mathbb{R}$ other than 0. More precisely, we state the following result on existence, uniqueness of strong solution and continuous dependence on initial data which holds for all $t \in \mathbb{R}$.

**Theorem 3.4.5.** *For $\mathbb{P}$-a.s. $\omega \in \Omega$, there hold*

- *For all $t_0 \in \mathbb{R}$ and all $v_0 \in H$, there exists a unique solution $v \in C([t_0,+\infty];H) \cap L^2_{loc}([t_0,+\infty);V)$ of equation (4.26) with initial value $v_0$.*
- *If $v_0 \in V$, then the solution belongs to $C([t_0,+\infty);V) \cap L^2_{loc}([t_0,+\infty);D(A))$.*
- *hence, for every $\varepsilon > 0, v(t) \in C([t_0+\varepsilon,+\infty);V) \cap L^2_{loc}([t_0+\varepsilon,+\infty);D(A))$.*
- *Denoting the solution by $v(t,t_0;\omega,v_0)$, then the map $v_0 \mapsto v(t,t_0;\omega,v_0)$ is continuous for all $t \geq t_0$, $v_0 \in H$.*

Theorem 3.4.5 implies the existence uniqueness of stochastic flow which is important for our study of RDS in the next chapter.

## 3.5 Invariant measures

The notion of invariant measure arises in Ergodic theory, which is an area concerns with the study of qualitative distributional properties of typical orbits of a dynamical system and that these properties are presented regarding measure theory. Roughly speaking, an invariant measure $\mu$ is a statistical stationary solution which represents the long-time behaviour of a given dynamical system. Moreover, if a given invariant measure $\mu$ can be proven to be unique, then it is possible that the probability law of the solution will converge to $\mu$. Hence, the unique invariant measure dictates the statistical equilibrium to which the system approaches give rise to the so-called ergodic measure. The rationale stems from the Birkhoff's Ergodic Theorem (see [7, 36]): given some time interval $[0,T]$, the time average on a time interval converges to the spatial average:

$$\frac{1}{T}\int_0^T P(t)\varphi dt = \frac{1}{T}\int_0^T \varphi(u(t))dt \to \int \varphi(u)\mu(du)$$

for all observables $\varphi$ of a given system and given initial condition $u_0$. Such a result has not been possible for the deterministic Navier-Stokes equation. This is what motivates one to study Navier-Stokes equation perturbed by noise.
The proof of the existence of invariant measure under conditions on the function and space is given by the Krylov-Bogolyubov argument.
Let $E$ be a Polish space. Let $\mathcal{E} = \mathcal{B}(E)$ be the sigma field of all Borel subset of $E$ and for $\Gamma \in \mathcal{E}$, let $I_\Gamma$ be the characteristic function

$$I_\Gamma(x) = \begin{cases} 1, & \text{if } x \in \Gamma \\ 0, & \text{if } x \in \Gamma^c, \end{cases}$$

where $\Gamma^c = E\backslash\Gamma$. Moreover Let $B_b(E)$, $C_b(E)$ denote the set of bounded Borel measurable (respectively bounded continuous function) and let $\mathcal{P}$ denotes the space of probability measures on $E$. Given $(u_t^x : x \in H, t \geq 0)$, a family of time-homogeneous $E$-valued Markov processes indexed by $x =: u_0^x$, define, $\forall\, t \geq 0$, $x \in E$, $\Gamma \in \mathcal{E}$, $(P_t)$, $(P_t^*)$ and $(P(t, x, \cdot))$ respectively the *markov semigroup* on $B_b(E)$, *adjoint Markov semigroup* on $\mathcal{P}$ and *transition* probability measures on $E$. More precisely, for any $t \geq 0$

$$P_t : \mathcal{B}(E) \to \mathcal{B}(E) \quad P_t f(x) = \mathbb{E} f(u_t^x) = \int_E f(y)P(t, x, dy), \quad f \in B_b(E),\ x \in E, \tag{3.163}$$

and

$$P_t^* : \mathcal{P} \to \mathcal{P}, \quad P_t^*\mu(\Gamma) = \int_E P(t, x, \Gamma)\mu(dx). \tag{3.164}$$

One can easily check that

$$\langle P_t f, \mu\rangle = \langle f, P_t^*\mu\rangle, \quad \forall\, f \in B_b(E),\ \mu \in B(E).$$

As a consequence of the definition of *transition* probability measures, one can define the markov transition functions as

$$P(t, x, \Gamma) = P_t 1_\Gamma(x) = \mathcal{L}(u_t^x), \quad t \geq 0, x \in E, \Gamma \in B(E). \tag{3.165}$$

To study asymtotic properties of the transition semigroup $P_t$, one has to study invariant measures. A Borel probability measure $\mu$ in $H$ is said to be invariant with respect to $P_t$ if

$$P_t^*\mu(\Gamma) := \int_E P_t(x, \Gamma)\mu(dx) = \mu(\Gamma), \quad \forall \Gamma \in \mathcal{B}(E), \tag{3.166}$$

or equivalently

$$\int_E f d\mu = \int_E P_t f d\mu \quad \forall t \geq 0, f \in B_b(E).$$

An important class of invariant measure is the ergodic measures.

**Definition 3.5.1.** Let $\mu$ be an invariant Borel measure on $E$ for $P_t$. A Borel set $\Gamma$ is said to be an invariant set with respect to semigroup $P_t$ if for any $t \geq 0$,

$$P_t 1_E = 1_E, \quad \mu \quad \text{a.s.}$$

The measure $\mu$ is said to be ergodic if $\mu(\Gamma)$ is either 0 or 1 for any invariant set $\Gamma$.

**Proposition 3.5.2.** *If $\mu$ is an invariant measure for $P_t$, then $\mu$ is ergodic if*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t \varphi dt = \bar{\varphi} \quad \textit{for all} \quad \varphi \in L^2(E, \mu),$$

*where*

$$\bar{\varphi} = \int_E \varphi(x) \mu(dx).$$

**Proof.** Let $\varphi = I_\Gamma$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t I_\Gamma dt = \bar{I}_\Gamma = \int_E I_\Gamma(x) \mu(dx) = \mu(\Gamma),$$

as $I_\Gamma$ is either 0 or 1.
Finally, as $I_\Gamma = \bar{I}_\Gamma$ can only be 0 or 1, then $\mu(\Gamma)$ can be either 0 or 1 and so $\mu$ is ergodic. □

The following theorem due to Doob is fundamental for the study of uniqueness of invariant measure [100].

**Theorem 3.5.3.** *Let $\mu$ be an invariant measure for the Markov family $(u^x, x \in E)$. If the corresponding semigroup $P_t$ is irreducible and strong feller, then $\mu$ is the unique invariant measure, and hence, ergodic.*

**Proof.** *Step 1 Prove* that the family of probability measures $(P_x(t, \cdot) : x \in E)$ are mutually equivalent for each $t > 0$.
This question boils down to how one proves two measures are equivalent, that is to show whether the two measures agree with an arbitrary set. If $\mu(A) = 0 = \nu(A)$, then clearly it does not distinguish the null set. Then if we have a set with positive probability starts at $x_0$, the

same should hold for any $x \in E$. With this in mind, one first needs to show for any $x_0 > 0$, if we let $\Gamma$ be such that, if we take any $x_0 \in E$ $P_{x_0}(t, \Gamma) > 0$.
To this end, pick any $t, s > 0$. Assume that $P_x(t + s, \Gamma) > 0$ for some $x \in E$, and $\Gamma \in \mathcal{E}$. By the property of the Chapman-Kolmogorov equation,

$$P_x(t+s, \Gamma) = \int_E P_x(t, dy) P_y(s, \Gamma).$$

Therefore, there exists a $y_0$ such that $P_{y_0}(s, \Gamma) > 0$. By the strong Feller property, $P_y(s, \Gamma)$ is a continuous function of $y$. Hence, there exists a neighborhood of $y_0$, $B_r(y_0)$ such that $P_y(s, \Gamma) > 0$ for all $y \in B_r(y_0)$. So, for any arbitrary $x \in E$,

$$P_x(t+s, \Gamma) = \int_E P_x(t, dy) P_y(s, \Gamma) \geq \int_{B_r(y_0)} P_x(t, dy) P_y(s, \Gamma) > 0,$$

since $P_x(t, B_r(y_0)) > 0$ and $P_y(s, \Gamma)$ for all $y \in B_r(y_0)$ by irreducibility. Thus $P_x(t + s, \Gamma) > 0$ for all $x \in E$.
*Step 2* Now let $\Gamma \in \mathcal{E}$ be an invariant set

$$P_t 1_\Gamma = 1_\Gamma, \forall\ t > 0$$

such that $\mu(\Gamma) > 0$. One has to show that $\mu(\Gamma) = 1$ By step 1, the family of probability measures $P_x(t, \cdot)$ indexed by $x$ are equivalent, so that $P_x(t, \Gamma) = 1$ for all $x \in E$. Moreover, by the invariance of $\mu$, it follows that

$$\mu(\Gamma) = \int_E P_x(t, \Gamma) d\mu(x) = 1.$$

Thus $\mu$ is ergodic. If $\nu$ is another invariant measure, then $\nu$ can be shown to be ergodic by repeating the above argument. Being invariant measures, both $\mu$ and $\nu$ are equivalent to $P_x(t, \cdot)$ for any $x$ and $t$. Therefore $\mu$ and $\nu$ are equivalent. Now it is well known that if $\mu$ and $\nu$ are two ergodic measures with respect to $P_t$ and if $\mu \neq \nu$, then $\mu$ and $\nu$ are singular. Hence $\mu = \nu$. ☐

In this section we are concerned with the existence of an invariant measure of the solution $u$ to the abstract equation (3.61) with the same assumptions used for Stokes operator $A$, bilinear operator $B$, operator $G$ and noise $L(t)$.
In the last section we proved existence and uniqueness of a weak solution Theorem 3.2.11, strong solutions Theorem 3.2.15 and the solution depends continuously on initial data Theorem 3.2.12. It is well known (see for instance Chapter 9 of [87]) that strong solution implies a weak solution, and the weak solution is equivalent to a mild solution. Hence the three concepts of solutions are equivalent. With the aid to these results, our main aim in this chapter is to study the large time behaviour of $u$, that is, the law $\mathcal{L}(u(t, x))$ as $t \to \infty$. In particular, we prove (3.61) admits at least one invariant measure Here we consider a general cadlag Markov process,

$$(\Omega, \{\mathcal{F}_t^0\}_{t \geq 0}, \mathcal{F}, \{u_t^x\}_{t \geq 0}, (\mathbb{P}_x)_{x \in H}) \tag{3.167}$$

whose transition probability is denoted by $\{P(t,x,dy\}_{t\geq 0}$, where $\Omega := D([0,\infty);H)$ is the space of the càdlàg function from $[0,\infty)$ to $H$ equipped with the Skorokhod topology, $\mathcal{F}_t^0 = \sigma\{u_s, 0 \leq s \leq t\}$ is the natural filtration. Now denote (resp.) by $C_b(H)$, $B_b(H)$ the space of bounded continuous functions and the space of bounded borel measurable functions on $H$. That is,

$$C_b(H) := \{\varphi : H \to \mathbb{R} : \varphi \text{ is continuous and bounded}\}, \tag{3.168}$$

$$B_b(H) := \{\varphi : H \to \mathbb{R} : \varphi \text{ is bounded and borel measurable}\}. \tag{3.169}$$

For all $\varphi \in B_b(H)$, define[4]

$$P_t\varphi(x) = \int_H \varphi(y)P(t,x,dy), \quad \forall\ t \geq 0, x \in H.$$

For any $t \geq 0$, $P_t$ is said to be Feller if

$$\varphi \in C_b(H) \to P_t\varphi \in C_b(H), \quad \forall \quad t \geq 0. \tag{3.170}$$

$P_t$ is said to be strong Feller if (3.170) holds for a larger class of function: $\varphi \in B_b(H)$. Moreover, $P_t$ is said to be *irreducible* in $H$, if $P_t 1_A(x) = P_x(t,A) > 0$ for any $x \in H$ and any non-empty open subset $A$ of $H$. If $P_t$ is irreducible then any invariant measure $\mu$ is full, that is, one has $\mu(B(x,r)) > 0$ for any ball $B(x,r)$ of center $x \in H$ and radius $r$. Indeed, it follows from (3.166) that

$$\mu(B(x,r)) = \int_H P_t 1_A(x)\mu(dx) > 0.$$

The main theorem proved in this section is Theorem 3.2.17, which we restate here for readers' convenience

**Theorem.** *Assume additionally, that there exists $m > 1$ such that $\sigma_l = 0$ for all $l \geq m$. Then the solution $u$ to (3.61) admits at least one invariant measure.*

We claim that the SNSE (3.61) has an invariant measure. The key to proving this is to use the Krylov-Bogolyubov Theorem (named after Russian-Ukrainian mathematicians and theoretical physicists Nikolay Krylov and Nikolay Bogolyubov), which guarantees the existence of invariant measures for certain well-defined maps defined on some well-defined space. More precisely, the theorem states that,

**Theorem 3.5.4** (Krylov-Bogolyubov)**.** *Assume $(P_t, t \geq 0)$ is a **Feller** semigroup. If there exists a point $x \in H$ for which the family of probability measure $\{\mu_t(x,\cdot)\}_{t\geq 0}$ is uniformly **tight**, that is, there exists a compact set $K_\epsilon \subset H$ such that $\mu(K_\epsilon) \geq 1-\epsilon$ for any $\mu \in \Lambda$ on $(H, \mathcal{B}(H))$ then there exists at least one invariant measure.*

[4]Alternatively, $P_t\varphi(x) = \mathbb{E}_x\varphi(u(t)) = E\varphi(u(t,x)) = \int \varphi(y)\mu_{x,t}(dy)$.

**Corollary 3.** *If for some $\nu \in \mathcal{P}$ and some sequence $T_n \uparrow +\infty$ the sequence $\{P^*_{T_n}\nu\}$ is tight, then there exists an invariant measure for $(P_t, t \geq 0)$.*

We shall remark that there are various versions of Krylov-Bogolyubov theorem which conveys the same idea. All that required to be proved are Feller, Markov property of the solution $v$ (and so $u$) and convergence of the family of probability measures $\{\mu_t, t \geq 0\}$ in $H$. This is comparable to the concept of weak convergence of distribution in finite dimension (equivalence to weak convergence of r.v.). However, in infinite dimension, the convergence of distribution is more involved. Hence extra conditions are needed besides the convergence of finite-dimensional distributions.

Note that, it is known that tightness is a necessary condition to prove convergence of probability measure, especially when measure space is infinite dimensional. In this sense the two statements of the theorem is equivalent.

The following inequalities would be used quite often.

$$|\hat{A}^\sigma e^{-\hat{A}t}| \leq C(\sigma)t^{-\sigma}, \quad \forall\ \sigma > 0,$$

$$|B(u)|_V = \langle A^{\frac{1}{2}}B(u), A^{\frac{1}{2}}B(u)\rangle = \langle AB(u), B(u)\rangle = |A^{\frac{1}{2}}B(u)| \tag{3.171}$$

$$\leq |u||u|_V^{\frac{1}{2}}|Au|^{\frac{1}{2}}, \tag{3.172}$$

$$|B(u) - B(v)| \leq C(|u|_V^2 + |u|_V^2|v| + |v|_V^2),$$

$$|B(u)| \leq C|u|_V^{\frac{1}{2}}|u|_V|Au|^{\frac{1}{2}} = C|u|_V^{\frac{3}{2}}|Au|^{\frac{1}{2}}.$$

### 3.5.1 Transition Semigroup

Let us denote by $u(\cdot, x)$ the solution of (3.61). We set

$$P_t f(x) = \mathbb{E}f(u(t,x)), \quad f \in B_b(H), t \geq 0, x \in H.$$

It follows from uniqueness and time homogeneity of $L$ that the following relation holds,

$$P_t \circ P_s = P_{t+s}.$$

Recall from our Theorem 3.2.16 that, we have proved there exists a unique strong solution to (3.61) with the form (3.76) in the space $D([0,T;H) \cap L^2([0,T];V)$ which belongs to $D([h,T];V) \cap L^2_{\text{loc}}([h,T];D(A))$ for all $h > 0$, $T > 0$ for every initial condition $u_0 \in H$, $\omega \in \Omega$. Moreover, if

$u_0 \in V$, then $u \in D([h,T;V]) \cap L^2_{\text{loc}}([h,T];D(A))$ for all $h > 0$ and $T > 0$. The solution depends continuously on initial data $x$.

Let $u(t;x)$ be the solution at $t$ starting from $x$ at time 0. Now suppose we have two solutions, resp. $u_n$ and $u$ of (3.61) started at $\xi_n$ and $\xi$, if the conditions in Theorem 3.2.12 satisfied, then it follows that $u_n(t) \to u(t)$ a.s. for any $t$. Therefore, $f(u_n(t)) \to f(u(t))$ as $f$ is continuous. Thus, invoke Lesbesgue Dominated Convergence Theorem, one has

$$\mathbb{E}f(u_n(t)) \to \mathbb{E}f(u(t)). \tag{3.173}$$

Whence the equation (3.61) defines a Feller Markov process. Then we can define the operator $P_t : C_b(H) \to C_b(H)$ by

$$P_t f = \mathbb{E}f(u(t;x)),$$

and $P_t$ is said to be a Feller semigroup.

**Lemma 3.5.5.** *The equation (3.61) defines a Markov process in the sense that*

$$\mathbb{E}[f(u^x_{t+s})|\mathcal{F}_t] = P_s(f)(u^x_t), \tag{3.174}$$

*for all $t, s > 0$, $f \in C_b(H)$, where $u^x_t$ denotes*[5] *the solution to (3.61) over $[0,\infty]$ starting from the point $u(0) = x$, $\mathcal{F}_t$ denotes the sigma-algebra generated by $L(\tau)$ for $\tau \leq t$.*

By uniqueness,

$$u^x_{t+s} = u^{u^x_t}_{t,t+s}, \quad (\mathbb{P} - a.s.),$$

where $(u^\eta_{t_0,t})_{t \geq t_0}$ denotes the **unique** solution on the time interval $[t_0, \infty)$, with the $\mathcal{F}_{t_0}$-measurable intial condition $u^\eta_{t_0,t_0} = \eta$. To prove (3.174), it suffices to prove

$$\mathbb{E}[f(u^\eta_{t,t+s})|\mathcal{F}_t] = P_s(f)(\eta),$$

for every $H$-valued $\mathcal{F}_t$-measurable r.v. $\eta$. Note that (3.174) holds for all $f \in C_b(H)$, holds for $f = 1_\Gamma$ where $\Gamma$ is an arbitrary Borel set of $H$ and consequently for all $f \in B_b(H)$. Without loss of generality, Let us assume $f \in C_b(H)$. We know that, if $\eta = \eta_i$ $\mathbb{P}$ a.s., then the r.v. $u(t+s, t, \eta_i)$ is independent to $\mathcal{F}_t$ and therefore

$$\mathbb{E}(f(u(t+s,t,\eta_i))|\mathcal{F}_t) = \mathbb{E}f(u(t+s,t,\eta_i)) = P_{t,t+s}f(\eta_i) = P_s f(\eta_i), \quad \mathbb{P} \text{ a.s.} \quad .$$

It suffices to prove (3.174) holds for every r.v. $\eta$ of the form

$$\eta = \sum_{i=1}^N \eta^{(i)} 1_{\Gamma^{(i)}},$$

[5] The notation $u^x_t$ is used interchangeably with $u(t;x)$.

where $\eta^{(i)} \in H$ and $\Gamma^{(i)} \subset \mathcal{F}_t$ is a partition of $\Omega$, $\eta_i$ are elements of $H$. Then

$$u(t+s,t,\eta_i) = \sum_{i=1}^{N} u(t+s,t,\eta_i) 1_{\Gamma_i}, \quad \mathbb{P} \text{ a.s.} \quad .$$

Hence,

$$\mathbb{E}(f(u(t+s,t,\eta))|\mathcal{F}_t) = \sum_{i=1}^{N} \mathbb{E}(f(u(t+s,t,\eta_i)) 1_{\Gamma_i}|\mathcal{F}_t) \quad \mathbb{P}-\text{a.s.} \quad .$$

Take into account the r.v. $u(t+s,t,\eta_i)$ independent to $\mathcal{F}_t$ and $1_{\Gamma_i}$ are $\mathcal{F}_t$ measurable, $i = 1,\cdots,k$, one deduces that

$$\mathbb{E}[f(u(t+s,t,\eta))|\mathcal{F}_t] = \sum_{i=1}^{N} P_s f(\eta_i) 1_{\Gamma_i} = P_s f(\eta), \quad \mathbb{P}-\text{a.s.} \quad ,$$

and so (3.61) defines a Markov process in the above sense for all $f \in C_b(H)$. Now, let $u(t;\eta)$ be the solution of the SNSE (3.61) with initial condiction $\eta \in H$.
Let $(P_t, t \geq 0)$ be the Markov Feller semigroup on $C_b(H)$ associated to the SNSE (3.61) defined as

$$P_t f(\eta) = \mathbb{E}[f(u(t;\eta))] = \int_H f(y) P(t,y) dy = \int_H f(y) \mu_{t,s}(dy), \quad f \in C_b(H), \tag{3.175}$$

where $P(t,x,dy)$ is the transition probability of $u(t;\eta)$ and $\mu_{t,x}(dy)$ is the law of $u(t;\eta)$. From (3.175), we have

$$P_t f(x) = (f, \mu_{t,x}) = (P_t f, \mu),$$

where $\mu$ is the law of the initial data $\eta \in H$. Thus it follows from above that $\mu_{t,\eta} = P_t^* \mu$. If

$$P_t^* \mu = \mu \quad \forall \quad t \geq 0,$$

then a probability measure $\mu$ on $H$ is said to be an invariant measure.

### 3.5.2 The proof of tightness

We proceed the claim of tightness by first proving the following a priori estimate. The main difficulty is overcome by introducing a simplified auxiliary Ornstein Uhblenbeck process, which enables us to use the classical arguments in the spirit of p.51-150 [1]. To prove existence of invariant measures for (3.61), we write the problem in a slightly different form.
Let $H$, $A : D(A) \subset H \to H$, $V = D(A^{1/2}) = D(\widehat{A}^{1/2})$ and $B : V \times V \to V'$, $C$ be spaces and operators introduced in the previous section. Suppose that there exists a constant $c_B > 0$ such that

$$\langle B(u,v), w\rangle = |b(u,v,w)| \leq c_B |u|^{1/2} |u|_V^{1/2} |v|^{1/2} |v|_V^{1/2} |w|_V, \quad \forall \quad u,v,z \in V, \tag{3.176}$$

$$\langle B(u,v), v\rangle \leq c_B |u|^{1/2} |Au|^{1/2} |v|_V |z|,$$

for all $u \in D(A)$, $v \in V$ and $z \in H$.

In order to prove there exists at least one invariant measure, we use standard method in the spirit of Chapter 15 in [36]. However, the analysis of Navier-Stokes equations with additive noise in our case requires some non-trivial consideration, as pointed out in [34],. In particular, a critical question arises when analyzing the estimate $\frac{d^+}{dt}|v(t)|^2$, the usual estimates for the nonlinear term $b(v(t), z(t), v(t))$ yields a term $|v(t)|^2 |z(t)|_4^4$, so we were not able to deduce any bound in $H$ for $|v(t)|^2$ under classical lines. Nevertheless, in light of the method developed in Crauel and Flandoli [34], via the usual change of variable and by writing the noise and the Ornstein-Uhlenbeck process as finite sequence of 1D processes, we are able to prove there exists at least one invariant measure to (3.177).

We remark that this fundamental ODE is different from the one used in the proof of existence and uniqueness. Let $f \in H$ and $m > 1$ be given. Consider

$$du(t) = [-Au(t) - B(u(t), u(t)) + \mathbf{C}u(t) + f]dt + \sum_{l=1}^{m} \sigma_l e_l dL_l(t), \tag{3.177}$$

where operators $A$, $B$, $C$ are defined earlier in this chapter (see also section 1.2), $f \in H$, $L_1$, $L_2 \cdots L_l$ are i.i.d. $\mathbb{R}$-valued symmetric $\beta$-stable process on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\sigma$ is a bounded sequence of real numbers and $e_l$ is the complete orthonormal system of eigenfunctions on $H$.

#### 3.5.2.1 Auxiliary Ornstein-Uhlenback Process

Let $(\tilde{L}(t), t \geq 0)$ be a Lévy process that is an independent copy of $L$. Denote by $\bar{L}$ a Lévy process on the whole real line by

$$\bar{L}(t) \begin{cases} L(t), & t \geq 0 \\ \tilde{L}(-t), & t < 0, \end{cases} \tag{3.178}$$

and by $\bar{\mathcal{F}}_t$ the filtration

$$\bar{\mathcal{F}}_t = \sigma(\bar{L}(s), s < t), \quad t \in \mathbb{R}.$$

Let $\alpha > 0$ be given; For each $l = 1, \cdots, m$, let $z_l^0$ be the stationary (ergodic) solution of the one dimensional equation

$$dz_l^0 + (\lambda_l + \alpha) z_l^0 dt = \sigma_l dL_l(t)$$

so that

$$z_l^0(t) = \int_{-\infty}^{t} e^{-(\lambda_l+\alpha)(t-s)}\sigma_l dL_l(s)$$

Note that the integral above is well defined, since for any $p \in (1,\beta)$ with $\beta > 1$ we have

$$\begin{aligned}\mathbb{E}\left|z_l^0(t)\right|^p &= C_{p,\beta}\int_0^{\infty} e^{-p(\lambda_l+\alpha)(t-s)}\sigma_l^p\,ds \\ &= \frac{C_{p,\beta}\sigma_l^p}{p\,(\alpha+\lambda_l)}.\end{aligned} \tag{3.179}$$

More precisely, let

$$z_l^0(t,s) = \int_s^t e^{-(\lambda_l+\alpha)(t-r)}\sigma_l dL_l(r)\,.$$

Then one can show directly evaluating integrals in the same way that

$$\lim_{s\to-\infty} z_l(t,s) = z_l(t)$$

exists. Putting $z^0(t) = \sum_{l=1}^m z_l^0(t)e_l$ one has

$$dz^0 + (A+\alpha I)z^0\,dt = GdL(t)\,, \tag{3.180}$$

where $Ge_l = \sigma_l e_l$, or

$$z^0(t) = \int_{-\infty}^{t} e^{-(t-s)(A+\alpha I)}GdL(s)\,.$$

We have for any $s,t$ such that $-\infty < s < t < \infty$

$$z(t) = \int_{-\infty}^{t} e^{-(t-s)\widehat{A}}GdL(s) = e^{-(t-s)\widehat{A}}z(s) + \int_s^t e^{-(t-r)\widehat{A}}GdL(r)\,.$$

We need another lemma.

**Lemma 3.5.6.** *We have*

$$\sup_{-1\le t\le 0} |Az(t)|^2 < \infty\,.$$

**Proof.** Note first that the process $Z^0 = Az^0$ is well defined and satisfies all the assumptions of Lemma 3.2.1 with the process $L$ replaced by another Lévy process $AL$. Therefore, we have

$$\sup_{-1\le t\le 0} \left|Z^0(t)\right|^2 < \infty\,.$$

Since $D(A) = D\left(\widehat{A}\right)$ all the rguments from the proof of Lemma 3.2.1 can be repeated yielding

$$\sup_{-1\le t\le 0} |Az(t)|^2 < \infty\,.$$

□

Now, using the lemma above and Lemma 3.2.1 applied with $\delta = \frac{1}{2}$ we find that the process $z$ is càdàg in $V$ and

$$\sup_{-1\le t\le 0}\left(|z(t)|^2 + |z(t)|_V^2 + |Az(t)|^2\right) < \infty \quad \mathbb{P} \text{ a.s.} \quad . \tag{3.181}$$

Using Proposition (3.2.3), one can now choose $\alpha > 0$ such that

$$4\eta m\mathbb{E}|z_1(0)| \le \frac{\lambda_1}{4}, \tag{3.182}$$

where $\lambda_1$ is the first eigenvalue of $A$, since $\mathbb{E}|z_1(0)|^p \to 0$ as $\alpha \to \infty$.
From (3.182) and the Ergodic Theorem we obtain

$$\lim_{t_0\to-\infty} \frac{1}{-1-t_0}\int_{t_0}^{-1} 4\eta\sum_{l=1}^{m}|z_l(s)|ds = 4\eta m\mathbb{E}|z_1(0)| < \frac{\lambda_1}{4}.$$

Put $\gamma(t) = -\frac{\lambda_1}{2} + 4\eta\sum_{l=1}^{m}|z_l(t)|$, we get

$$\lim_{t_0\to-\infty} \frac{1}{-1-t_0}\int_{t_0}^{-1}\gamma(s)ds < -\frac{\lambda_1}{4}. \tag{3.183}$$

From this fact and by stationarity of $z_l$ we finally obtain

$$\lim_{t_0\to-\infty} e^{\int_{t_0}^{-1}\gamma(s)ds} = 0 \quad \mathbb{P}-\text{a.s.} \quad , \tag{3.184}$$

$$\sup_{t_0<-1} e^{\int_{t_0}^{-1}\gamma(s)ds}|z(t_0)|^2 < \infty, \quad \mathbb{P}-\text{a.s.} \quad . \tag{3.185}$$

$$\int_{-\infty}^{-1} e^{\int_{\tau}^{-1}\gamma(s)ds}(1 + |z_l(\tau)|^2 + |z_l(\tau)|_V^2 + |z_l(\tau)|^2|z_l(\tau)|)d\tau < \infty, \quad \mathbb{P}-\text{a.s.} \quad . \tag{3.186}$$

for all $1 \le j, l \le m$. Indeed, note for instance that for $t < 0$,

$$\frac{z_l(t)}{t} = \frac{z_l(0)}{t} - \frac{1}{t}(\alpha + A_l)\int_t^0 z_l(s)ds + \frac{L_l(t)}{t},$$

whence $\lim_{t\to-\infty}\frac{z_l(t)}{t} = 0$ $\mathbb{P}$-a.s., which implies (3.184) and (3.185). Consider the abstract SNSE

$$du + [Au + B(u) + \mathbf{C}u]dt = fdt + GdL(t)$$

and the Ornstein-Uhlenback equation

$$dz + (\hat{A} + \alpha I)zdt = GdL(t),$$

where $L(t) = \sum_{l=1}^{m} e_l L_l(t)$. We now use the change of variable $v(t) = u(t) - z(t)$. Then, by subtracting the Ornstein-Uhlenback equation from the abstract SNSE, we find that $v$ satisfies the equation

$$\frac{d^+ v}{dt} = -\nu A v(t) - \mathbf{C} v(t) - B(u,u) + f + \alpha z. \tag{3.187}$$

Recall the Poincare inequalities

$$|u|_V^2 \geq \lambda_1 |u|^2, \quad \forall \quad u \in V, \tag{3.188}$$

$$|Au|^2 \geq \lambda_1 |u|^2, \quad \forall \quad u \in D(A). \tag{3.189}$$

Let us note that there exists $\eta > 0$ such that

$$|\langle B(u, e_l), u\rangle| \leq \eta |u|^2, \quad u \in V, l = 1, \cdots, m. \tag{3.190}$$

Then the following holds.

**Proposition 3.5.7.** *Let $\alpha > 0$, $v$ is a mild solution of (3.187), there exist cosntants $c, c' > 0$ depending only on $\lambda_1$ such that*

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \leq \left(-\frac{\lambda_1}{4} + 2\eta \sum_{l=1}^m |z_l|\right)|v|^2 + c|f|^2 + c\alpha|z|^2 + 2\eta|z|^2 \sum_{l=1}^m |z_l|. \tag{3.191}$$

**Proof.** Let $\alpha > 0$ be given. Denote for simplicity by $z(t)$ the stationary Orstein Uhbleck process, corresponding to $\alpha$, introduced in earlier. Using the classical change of variable $v(t) = u(t) - z(t)$ , the well known identity $\frac{1}{2}\partial_t |v(t)|^2 = (v(t), v(t))$, and the antisymmetric term $(Cv, v) = 0$ we have

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 = -\nu(Av, v) - \langle B(u,z), u\rangle + (\alpha z, v) + \langle f, v\rangle \tag{3.192}$$

$$\leq -\nu|v|_V^2 - \langle B(u,z), u\rangle + \alpha|z||v| + |f||v|. \tag{3.193}$$

By the definition of $z$ and assumptions (3.190),

$$\langle B(u,z), u\rangle = \sum_{l=1}^m \langle B(u, e_l), u\rangle z_l \leq \eta |u|^2 \sum_{l=1}^m |z_l|$$

$$\leq 2\eta|v|^2 \sum_{l=1}^m |z_l| + 2\eta|z|^2 \sum_{l=1}^m |z_l|,$$

and the inequalities

$$\langle \alpha z, v\rangle = c\alpha|z|^2 + c'|v|^2,$$

$$\langle f, v\rangle \leq c|f|^2 + c'|v|^2.$$

For simplicity we take $\nu = 1$. Then via Young inequality, one can show that there exists $c, c' > 0$ depending only on $\lambda_1$ such that

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \le -\frac{1}{2}|v|^2 + 2\eta|v|^2\sum_{l=1}^m |z_l| + 2\eta|z|^2\sum_{l=1}^m |z_l| + c|f|^2 + 2c'|v|^2 + c\alpha|z|^2 + 2c|z|_V^2 + c'|v|_V^2.$$

So

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \le (-\frac{\lambda_1}{4} + 2\eta\sum_{l=1}^m |z_l| + 2c')|v|^2 + c|f|^2 + c\alpha|z|^2 + 2\eta|z|^2\sum_{l=1}^m |z_l|.$$

Hence one can find a constant $c, c' > 0$ depending only on $\lambda_1$ for which the claim follows. Moreover, Let $\gamma(t)$, and $p(t)$ are defined as :

$$p(t) = c|f|^2 + c\alpha|z|^2 + \eta|z|^2\sum_{l=1}^m |z_l(t)|,$$

$$\gamma(t) = -\frac{\lambda_1}{2} + 4\eta\sum_{l=1}^m |z_l(s)|,$$

we have

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \le \frac{1}{2}\gamma(t)|v|^2 + p(t). \tag{3.194}$$

□

Temporarily disregard the $|v(t)|_V$ term, we have

$$\frac{d^+}{dt}|v(t)|^2 \le \gamma(t)|v(t)|^2 + 2p(t)$$

which implies

$$|v(t)|^2 \le |v(\tau)|^2 e^{\int_\tau^t \gamma(s)ds} + \int_{t_0}^t e^{\int_s^t \gamma(\xi)d\xi} 2p(s)ds. \tag{3.195}$$

Now drop out the first term in (3.194), integrate over $[\tau, t]$ we have

$$\int_\tau^t |v(s)|_V^2 ds \le (\sup_{\tau\le s\le t} |v(s)|^2)\int_\tau^t \gamma(\xi)d\xi + \int_\tau^t 2p(s)ds. \tag{3.196}$$

Let us recall, that we proved the existence an uniqueness of solutions to the stochastic Naver-Stokes equation under the assumption that

$$\sum_{l=1}^\infty |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty\,, \tag{3.197}$$

and then the process $z(\cdot)$ has a càdlàg version in $V = D(A^{1/2})$.
We will show that, under the above assumption, there exist at least one invariant measure for SNSE. Let $f \in H$, be given. For an arbitrary real number $s$, $u(t, s)$, $t \ge s$, is the unique solution

to the SNSE

$$\begin{cases} du(t) + Au(t)dt + B(u(t), u(t))dt + \mathbf{C}u(t)dt = fdt + dL(t), \\ u(s) = 0 \end{cases}$$

**Remark.** The space $D(A^\delta)$ is compactly embedded into the space $H$.

Consequently, if one prove that the process $u(t,0)$, $t \geq 1$ is bounded in probability as a process with values on $D(A^\delta)$, one gets immediately the law $\mathcal{L}(u(t,0))$, $t \geq 1$ are tight on $H$. This suffices the claim of existence of an invariant measure. More precisely, one proves in two steps.
**Step 1** Assuming that (3.197) holds we will prove an a priori bound in $H$. For any $\alpha \geq 0$, denote by $z$ the stationary solution of

$$dz_\alpha + (\hat{A} + \alpha I)z_\alpha dt = d\bar{L}(t),$$

where

$$z_\alpha(t) = z(t) + e^{-(\hat{A}+\alpha)(t-s)}(z_\alpha - z(s)) - \alpha \int_s^t e^{-(\hat{A}+\alpha)(t-s)} z(\sigma)d\sigma. \tag{3.198}$$

Let

$$v_\alpha(t,s) = u(t,s) - z_\alpha(t), \quad t \geq s.$$

Then $v_\alpha(t) = v_\alpha(t,s)$, $t \geq s$ is the mild solution to

$$\begin{aligned} \partial_t v_\alpha(t) + \nu A v(t)dt + \mathbf{C}(v_\alpha(t) + z_\alpha(t)) &= -B(v_\alpha(t) + z_\alpha(t)) + f + \alpha z_\alpha(t), \quad t \geq s, \\ v(s) &= -z_\alpha(s) \end{aligned}$$

Following step 1 one has the following proposition

**Proposition 3.5.8.** *There exists $\alpha > 0$ and a random variable $\xi$ such that $\mathbb{P}$-a.s.*

$$|v_\alpha(t,s)| \leq \xi \quad \forall\, t \in [-1,0] \quad \textit{and all} \quad s \leq -1, \tag{3.199}$$

$$\int_{-1}^0 |v_\alpha(t,s)|_V^2 ds < \xi \quad \forall\ t \in [-1,0] \textit{ and all } s \leq -1. \tag{3.200}$$

**Proof.** In view of inequality 3.195, one obtains

$$|v_\alpha(t,s)|^2 \leq |v(s)|^2 e^{\int_s^t -\frac{\lambda_1}{2} + 4\eta \sum_{l=1}^m |z_l(\xi)| d\xi} + \int_s^t e^{\int_r^t \gamma(\xi)d\xi} 2p(r)dr. \tag{3.201}$$

Based on the earlier discussion, the first term is finite; The second term is also finite under the assumption (3.181).

We now use the ergodic properties of $z$. Since $z_\alpha(t)$, $-\infty < t < \infty$, is an ergodic process which is supported by $D(A^\delta) \subset \mathbb{L}^4(\mathbb{S}^2)$. Then by the Marcinkiewicz strong law of large number, we have $\mathbb{P}$ a.s. that and by Prop 8.4 [23] that

$$\lim_{s\to-\infty} \frac{1}{-1-s}\int_s^{-1} 4\eta \sum_{l=1}^{m} |z_l(\sigma)| d\sigma = 4\eta m \mathbb{E}|z_1(0)| < \frac{\lambda_1}{4}.$$

The existence and uniqueness of invariant measure for the OU equation driven by Lévy process is well-known [29].

Let $\mu_\alpha$ be the unique invariant measure of Lévy type. It is easy to see that

$$\lim_{\alpha\to\infty} \int_V 4\eta \sum_{l=1}^{m} |z_l(s)| \mu_\alpha(dz) = 0.$$

Then for sufficiently large random $s_0 > 0$ and $s < -s_0$

$$e^{\int_s^t -\frac{\lambda_1}{2} + 4\eta\sum_{l=1}^{m} |z_l(\xi)| d\xi} \le e^{-\frac{\lambda_1}{4}(t-s)}. \tag{3.202}$$

To complete the proof this proposition we need the following Lemma.

**Lemma 3.5.9.** *Assume that $X$ is a stationary process taking values in a Banach space $B$. Moreover, assume that for a certain $p > 0$ we have*

$$\mathbb{E} \sup_{t\in[-1,0]} |X(t)|_B^p < \infty \,.$$

*Then for every $\kappa > 0$ such that $\kappa p > 1$ there exists a random variable $\xi$ such that $\mathbb{P}$ a.s.*

$$|X(t)|_B \le \xi + 2^\kappa |t|^\kappa,$$

*for all $t \le 0$.*

**Proof.** Let $\eta_n = \sup_{-n\le s\le -n+1} |X(s)|_B$, $n = 0, 1, \ldots$ Then by stationarity

$$\mathbb{E}\eta_n^p = \mathbb{E}\eta^p < \infty.$$

Therefore,

$$\mathbb{P}(\eta_n \ge n^\kappa) \le \frac{\mathbb{E}\eta_1^p}{n^{\kappa p}}. \tag{3.203}$$

If $\kappa p > 1$, then $\sum_{n=1}^\infty \mathbb{P}(\eta_n \ge n^\kappa) < \infty$, and by the Borel Cantelli lemma, $\mathbb{P}$-a.s., for any sufficiently large $n$,

$$\eta_n \le n^\kappa.$$

That is, for every $\omega$, there exists $N(\omega)$ such that

$$\eta_n(\omega) \le n^\kappa, \quad \text{for} \quad n > N(\omega).$$

Therefore, for $t \in [-n, -n+1]$ we have

$$|X(t,\omega)|_B \leq \eta_n(\omega) \leq \eta_n(\omega) I_{n \leq n(\omega)} + n^\kappa I_{n > N(\omega)}$$
$$\leq \eta_n(\omega) I_{n \leq N(\omega)} + 2^\kappa |t|^\kappa.$$

Since $\mathbb{P}(N < \infty) = 1$ the random variable

$$\xi(\omega) = \max_{n \leq N(\omega)} \eta_n(\omega)$$

is finite $\mathbb{P}$-a.s. and the Lemma follows. □

With the aid of this lemma, combine with equations (3.199), (3.201) and (3.202). We deduce the claim in Proposition 3.5.8. Moreover, via an apriori estimate about $\int_0^T |v(t)|_V^2 dt$, that is (3.196) the inequality (3.200) follows. □

**Step 2** *Measure support.* We now generalise Proposition 3.5.7 by proving regularizing property of equation (3.187) via deducing a priori estimate in $D(A^\delta)$ for some $\delta > 0$. This allows us to establish support of invariant measure.

**Proposition 3.5.10.** *For any $\delta \in [0, \frac{1}{2}]$, there exists $C = C(\delta)$ such that for any mild solution $v(\cdot)$ of (3.187), one has*

$$|A^\delta v(t)|^2 \leq e^{C\int_0^t |v(s)|^2 |A^{\frac{1}{2}} v(s)|^2 ds} |A^\delta v(0)|^2 + C \int_0^t e^{C\int_\sigma^t |v(s)|^2 |A^{\frac{1}{2}} v(s)|^2 ds} (|A^{\delta+\frac{1}{2}} f|^2 + |z(\sigma)|^2 + |A^{\frac{1+2\delta}{4}} z(\sigma)|^4) d\sigma. \tag{3.204}$$

**Proof.** Multiply (3.187) by $A^{2\delta} v$ and integrating over $\mathbb{S}^2$, one finds that

$$\begin{aligned} &\frac{1}{2}\partial_t |A^\delta v(t)|^2 + |A^{\frac{1}{2}+\delta} v(t)|^2 + (\mathbf{C} v(t), A^{2\delta} v(t)) \\ = &-b(v(t) + z_\alpha(t), v(t) + z_\alpha(t), A^{2\delta} v(t)) + \alpha(A^\delta z_\alpha(t), A^\delta v(t)) + \langle A^\delta f, A^\delta v(t)\rangle. \end{aligned} \tag{3.205}$$

From Lemma 3.1.4 it is clear that

$$(\mathbf{C} v, A^{2\delta} v) = 0.$$

To complete the proof we need to estimate the terms $b(v+z, v+z, A^{2\delta} v)$, $\alpha\langle A^{2\delta} v, z\rangle$, $\langle A^{2\delta} v, f\rangle$ Using Young inequality with $ab = \sqrt{\frac{\nu}{10}} a \sqrt{\frac{10}{\nu}} b$, $p = 2$, we have

$$\alpha|\langle A^{2\delta} v, z\rangle| \leq \frac{\nu}{6} |A^{\delta+\frac{1}{2}} v|^2 + \frac{3\alpha^2}{2\nu} |z|^2$$

$$|\langle A^{2\delta} v, f\rangle| \leq \frac{\nu}{6} |A^{\delta+\frac{1}{2}} v|^2 + \frac{3}{2\nu} |f|^2$$

Finally, following the method of deriving (15.4.12) as in [36], one can show that, for any $\nu > 0$, there exists a $K(\nu)$ such that

$$|\langle A^{2\delta}v, B(v+z, v+z)\rangle| = |b(v+z, v+z, A^{2\delta}v)| \tag{3.206}$$

$$\leq \frac{\nu}{6}|A^{\delta+\frac{1}{2}}v|^2 + K(\nu)(|v|^2|A^{1/2}v|^2 + |A^{\frac{1+2\delta}{4}}z|^4). \tag{3.207}$$

Combing the above estimates, we have

$$\frac{1}{2}\partial_t|A^\delta v(t)|^2 + (1-3\frac{\nu}{6})|A^{\frac{1}{2}+\delta}v(t)|^2$$
$$\leq K(\nu)|v(t)|^2|A^{\frac{1}{2}}v(t)|^2 + K(\nu)|A^{\frac{1+2\delta}{4}}z(t)|^4 + \frac{3\alpha^2}{2\nu}|z(t)|^2 + \frac{3}{2\nu}|f|^2.$$

Therefore, invoking Gronwall, it follows that

$$|A^\delta v(t)|^2 \leq e^{K(\nu)\int_0^t |v(s)|^2|A^{\frac{1}{2}}v(s)|^2 ds}|A^\delta v(0)|^2 \tag{3.208}$$

$$+ \int_0^t e^{K(\nu)\int_\sigma^t |v(s)|^2|A^{\frac{1}{2}}v(s)|^2 ds}\left(\frac{3\alpha^2}{2\nu}|z|^2 + K(\nu)|A^{\frac{1+2\delta}{4}}|^4 + \frac{3}{2\nu}|f|^2\right)d\sigma. \tag{3.209}$$

□

To complete the proof of invariant measure. It follows from Proposition 3.5.10 that for any $t \leq -1 \leq r \leq 0$,

$$|A^\delta v_\alpha(0,t)|^2$$
$$= e^{K(\nu)\int_r^0 |v_\alpha(s,t)|^2|A^{\frac{1}{2}}v_\alpha(s,t)|^2 ds}|A^\delta v_\alpha(r,t)|^2$$
$$+ \int_r^0 e^{K(\nu)\int_\sigma^0 |v_\alpha(s,t)|^2|A^{\frac{1}{2}}v_\alpha(s,t)|^2 ds}(\frac{3\alpha^2}{\nu}|z|^2 + K(\nu)|A^{\frac{1+2\delta}{4}}|^4 + \frac{3}{2\nu}|f|^2)d\sigma$$
$$\leq e^{K(\nu)[\sup_{-1\leq s\leq 0}|v_\alpha(s,t)|^2]\int_{-1}^0 |A^{\frac{1}{2}}v_\alpha(s,t)|^2 ds} \times \left[|A^\delta v_\alpha(r,t)|^2 + \frac{3\alpha^2}{2\nu}|z|^2 + K(\nu)|A^{\frac{1+2\delta}{4}}|^4 + \frac{3}{\nu}|f|^2 d\sigma\right].$$

Consequently, integrating the above over the interval $[-1,0]$, one gets for $t \leq -1$ that

$$|A^\delta v_\alpha(0,t)|^2$$
$$\leq e^{K(\nu)[\sup_{-1\leq s\leq 0}|v_\alpha(s,t)|^2]\int_{-1}^0 |A^{\frac{1}{2}}v_\alpha(s,t)|^2 ds}$$
$$\times \left[|A^{\frac{1}{2}}v_\alpha(r,t)| + \frac{3\alpha^2}{2\nu}|z|^2 + K(\nu)|A^{\frac{1+2\delta}{4}}|^4 + \frac{3}{2\nu}|f|^2 d\sigma\right].$$

By Proposition 3.5.8 there exists a random variable $\eta$ such that $\mathbb{P}$ a.s.

$$|A^\delta v_\alpha(0,t)| \leq \xi, \quad \forall\ t \leq -1. \tag{3.210}$$

Moreover,

$$|A^\delta u(0,t)| \leq |A^\delta v_\alpha(0,t)| + |A^\delta z_\alpha(0)|.$$

Since $z_\alpha(0)$ takes value in $D(A^\delta)$ there exists another random variable $\zeta$ such that $\mathbb{P}$ a.s.

$$|A^\delta u(0,t)| \leq \zeta \quad \forall\ t \leq -1. \tag{3.211}$$

So $u(0,t)$ is bounded in probability in the space $D(A^\delta)$ for some $\delta > 0$ satisfies $\sum_{l\geq 1} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty$:

$$\forall\ \epsilon > 0\ \exists\ R > 0\ \forall\ t \geq 0 \quad \mathbb{P}(|u(0,t,u_0)| \geq R) < \epsilon.$$

Now Let $u_0$ be fixed and let $\nu_{t,u_0}$ be the law of $u(t,u_0)$. Set

$$\mu_T = \frac{1}{T}\int_0^T \nu_{t,u_0} dt.$$

Let $B_R = \{x \in D(A^\delta); |A^\delta x| \leq R\}$, equation (3.211) implies for $p \in (1,\beta)$

$$\begin{aligned}\mu_T(B_R^c) &\leq \frac{1}{TR^p}\int_0^T \mathbb{E}|A^\delta u(0,t,u_0)|^p\, dt \\ &\leq \frac{1}{TR^p} T\mathbb{E}\zeta^p = \frac{\mathbb{E}\zeta^p}{R^p}.\end{aligned}$$

We have that, for any $\epsilon > 0$, $\mu_T(B_R) = 1 - \epsilon$ for sufficient large $R$. Hence $\mu_T$ is tight and its limit is an invariant probability measure of the solution $u$ of equation (3.61), by Corollary 3. Moreover, the support of the invariant measure is in $D(A^{1/2})$.
Combine with the markov feller properties proved for $u$ earlier, the solution $u$ to equation (3.61) admits at least one invariant measure and is supported in $D(A^{1/2})$. Hence, Theorem 3.2.17 is proved.

## 3.6 Appendix

### 3.6.1 Miscellaneous facts from Functional Analysis

In this section we review some basic facts from Functional Analysis. For the sake of brevity proofs are omitted. Readers are referred to the appendix in [92]. These well known facts also appear in standard texts such as [13, 77, 116].

#### 3.6.1.1 Weak and weak* convergence and compactness

Suppose $U$ is a Banach space (over $\mathbb{R}$) then its *dual space* $U^* := \mathcal{L}(U,\mathbb{C})$ consists of all bounded linear functionals from $U$ to $\mathbb{R}$. The dual space is a Banach space, where the norm of $f \in U^*$ is

$$\|f\| = \sup_{x\in U:\|x\|=1} \langle f, x\rangle_{U\times U^*} = \sup_{x\in U:\|x\|=1} |f(x)|.$$

The subscript $U \times U^*$ from the dual pairing is often omited.

Let us now recall some basic fact concerning dual spaces, weak and weak* convergence. For proofs see for instance Yosida [116].
A fundamental result is the Hahn-Banach Theorem, which allows us to extend linear functionals defined on subspaces to the whole space.

**Theorem 3.6.1** (Hahn-Banach Theorem)**.** *Let $U$ is a Banach space and $E$ be a linear subspace of $U$. If $f \in E^*$ then there exists an $F \in U^*$ that satisfies $F(u) = f(u)$ for every $u \in E$ and $|F|_{U*} \leq |f|_{E^*}$.*

**Corollary 4.** *Let $U$ be a Banach space and take $u \in U$ with $u \neq 0$. Then there exists $f \in U^*$ such that $|f|_{U^*} = 1$ and $\langle f, u\rangle = |u|_U$.*
A sequence $(x_n)$ in $U$ converges weakly to $u \in U$, which we write as $u_n \rightharpoonup u$ in $U$, if

$$\langle f, u_n\rangle \to \langle f, u\rangle \quad \text{for every} \quad f \in U^*.$$

The following is a list of properties of weak limits.

**Proposition 3.6.2.** *Let $U$ be a Banach space and $(u_n)$, $u \in U$.*

- *Weak limits are unique;*
- *If $u_n \to u$ then $u_n \rightharpoonup u$;*
- *weak convergence sequences are bounded; and*
- *if $u_n \rightharpoonup u$ then*

$$|u|_U \leq \lim \inf_{n\to\infty} |u_n|_U,$$

*so in particular if $|u_n|_E \leq M$ for every $n$ it follows that $|u|_E \leq M$.*

**Lemma 3.6.3.** *If $H$ is a Hilbert space and $(u_n) \in H$ then $u_n \to u$ if and only if $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$*
A sequence $(f_n)$ in $U^*$ converges weakly* to $f \in U^*$, which we write as $u_n \overset{*}{\rightharpoonup} u$ in $U^*$, if

$$\langle f_n, u\rangle \to \langle f, u\rangle \quad \text{for each} \quad u \in U.$$

One may observe that this is in some sense the natural definition of convergence in $U^*$. Moreover, weak* convergence satisfies the following properties.

**Proposition 3.6.4.** *Let $U$ be a Banach space and $(f_n)$, $f \in U^*$.*

*(i) Weak* limits are unique;*

*(ii)* *weak** *convergent sequences are bounded; and*

*(iii)* *if* $f_n \overset{*}{\rightharpoonup} f$ *then*

$$|f|_{U^*} \leq \liminf |f_n|_{U^*},$$

*so in particular if* $|f_n|_U^*$ *for every* $n$ *it follows that* $|f|_{U^*} \leq M$.

The following two results provide weak and weak* compactness and are central to the arguments that construct solutions as limits of approximations.

**Theorem 3.6.5** (Banach-Alaoglu Theorem)**.** *If* $U$ *is a separable Banach space then any bounded sequence in* $U^*$ *has a weakly** *convergent subsequence.*

**Corollary 5** (Weak compactness)**.** *Let* $U$ *be a reflexive Banach space, then any bounded sequence* $u_n$ *has a convergent subsequence, that is, there exists* $u_{n_k} \subset u_n$ *and* $u \in U$ *such that* $u_{n_k} \rightharpoonup u$

#### 3.6.1.2 Some convergence theorems

Let $\mu$ be a measure on $\mathbb{R}^d$.

**Theorem 3.6.6** (Dominated convergence theorem)**.** *Let* $(f_n)$ *be a sequence of measurable function on* $\mathbb{R}^d$ *such that*

- $f_n \to f$ almost everywhere *(equivalently,* $\lim_{n\to\infty} f_n(x) = f(x)$ *pointwise.)*
- *there exists nonnegative integrable function* $g$*, i.e.* $g : \mathcal{S} \to [0, \infty]$ *such that* $|f_n| \leq g$ *a.e.* $\forall\, n \in \mathbb{R}^d$*. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f| d\mu = 0,$$

*in particular*

$$\int f_n \to \int f.$$

**Lemma 3.6.7** (Fatou)**.** *Let* $(f_n)$ *be a sequence of nonnegative measurable function on* $\mathbb{R}^d$ *(i.e.* $f_n : \mathcal{S} \to [0, \infty]$ *is an arbitrary measurable function) such that*

- $f_n \to f$ *a.e.*
-

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n.$$

**Theorem 3.6.8** (D.C.T. Continuity of integral)**.** *Let $\mu$ be a measure on $X$ and $I \subseteq \mathbb{R}^d$. Suppose that $F : X \times I \to \mathbb{R}^d$ is such that*

- *For each $t \in I$, the function $F(\cdot, t)$ is $\mu$-integrable;*
- *For each $t \in I$, $F(x, t)$ is coninuous at $t_0$ for almost all $x \in \mathbb{R}^d$*
- *there exists $g \in L^1(X; \mathbb{R}^d)$ such that $|F(t, x)| \leq G(x)$ for almost all $x \in X$ and all $t \in I$*

*Define $f(t) = \int_X F(t, x) d\mu(x)$. Then $f$ is continuous at $t_0 \in I$.*

**Theorem 3.6.9** (Differentiation under the integral)**.** *Let $\mu$ be a measure on $X$ and $I \subseteq \mathbb{R}^d$. Suppose that $F : X \times I \to \mathbb{R}^d$ is such that*

- *For each $t \in I$, the function $F(\cdot, t)$ is $\mu$-integrable;*
- *For each $t \in I$, $D_t F(x, t)$ exists for* almost all $x \in X$;
- *there exists $g \in L^1(\mu)$ such that $\sup_{t \in I} |D_t F(x, t)| \leq g(x)$ for almost all $x \in X$.*

*Define $f(t) = \int_{\mathbb{R}^d} F(t, x) d\mu(x)$. Then $f$ is differentiable and*

$$f'(t) = \int_{\mathbb{R}^d} DF(t, x) d\mu(y),$$

*where*

$$D_t F(t, x) = \frac{\partial F}{\partial t} = \lim_{h \to 0} \frac{F(t + h, x) - F(t, x)}{h}$$
$$D_x F(t, x) = \frac{\partial F}{\partial x} = \lim_{h \to 0} \frac{F(t, x + h) - F(t, x)}{h}$$

### 3.6.2 Limit theorems

The law of large numbers plays a cenral role in probability and statistics. Roughly speaking, it says, if one repeats an experiment independenly many many times and average the result, then one obtain a value close to the expected value. Here we review three main limit theorems. Namely, WLLN, SLLN and the Marcinkiewicz Strong Law of Large Numbers.

**Theorem 3.6.10** (Weak Law of Large Numbers (WLLN))**.** *Let $\xi_1, \xi_2, \cdots, \xi_n$ be i.i.d. r.v. with finite first moment: $\mathbb{E}\xi = \mu < \infty$. Then,*

$$\lim_{n \to \infty} \left( |\bar{\xi}_n - \mu| \geq \epsilon \right) = 0.$$

*Remark*: $\bar{\xi}_n$ is said to converges in probability to $\mathbb{E}\xi$, denotes at $\xi_n \xrightarrow{\mathbb{P}} \mu$.

**Theorem 3.6.11** (Kolmogorov Strong Law of Large Numbers (SLLN)). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be i.i.d. r.v. with finite first moment: $\mathbb{E}\xi = \mu < \infty$. Then,*

$$\mathbb{P}(\omega : \lim_{n\to\infty} \bar{\xi}_n(\omega) = \mu) = 1.$$

*Remark*: $\bar{\xi}_n$ is said to converges to $\mu$ almost surely , denotes at

$$\xi_n \xrightarrow{\text{a.s.}} \mu \Leftrightarrow \sup_{k\geq n} |\bar{\xi}_k - \mu| \xrightarrow{\mathbb{P}} 0 \implies \bar{\xi}_n \xrightarrow{\mathbb{P}} \xi.$$

Roughly speaking, take $S_n = \xi_1 + \cdots + \xi_n$, then the WLLN states that, for every sufficiently large fixed $n$ the average $S_n/n$ is likely to be close to $\mu$. The SLLN on the other hand ask the question that, in what sense can we say

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = \mu.$$

Now we introduce a version of SLLN applicable to the $\beta$-stable case.

**Theorem 3.6.12** (Marcinkiewicz SLLN). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be i.i.d. r.v.. If $0 < p < 2$ then the relation $\mathbb{E}|\xi_1|^p < \infty$ is equivalent to the relation*

$$\frac{S_n - n\mu}{n^{1/p}} \to 0 \quad a.s. \quad .$$

*Here, $\mu = 0$ if $0 < p < 1$ and $\mu = \mathbb{E}\xi_1$ if $1 \leq p < 2$.*

# CHAPTER 4

## Random dynamics, Random attractors and Invariant measures

### 4.1 Introduction and Motivation

The study of asymptotic behaviour of dynamical systems has been one of the most fundamental problems in mathematical physics. One of the central notions is an attractor, which conveys crucial geometric information about the asymptotic regime of a dynamical system as $t \to \infty$. It is well known that the 2D Navier-Stokes equations is dissipative and so have a global attractor, see for instance [91, 103]. More precisely, there exists compact subset $K$ of the original phase space where all asymptotic dynamics fall in. Much of the theory of infinite dynamical systems devote to study the properties of this set $K$, which is called the *global attractor* (see for instance [91, 103]). For instance, one can show that, if there exists a unique mild solution on the space $K$ then we can define a group of solution operator $S(t)$ for the equation sensibly for all $t \in \mathbb{R}$, this defines a standard *dynamical system*,

$$(K, \{S(t)_{t\in\mathbb{R}}\})$$

A random (pullback) attractor is the pullback attractor where time-dependent forcing become random (In this thesis, this is given by the contribution from $F(t)$ and the noise term ). Readers are referred to [28] for a comparison of the three frameworks for the study of attractors, namely, attractors, pullback attractors and random attractors. Just like in the deterministic case, the theory of random attractor plays an important role in the study of the asymptotic behaviour of dissipative random dynamical system. Crauel and Flandoli [34], Crauel, Debussche and Flandoli [33] developed a theory for the existence of random attractors for stochastic systems that closely comparable to the deterministic theory. Roughly speaking, a random attractor is a random invariant compact set which attracts every trajectory as time goes to infinity. The strategy to prove the existence of random attractor is analogous to the method of proving global attractor in deterministic case, which involves two main methods. The first method requires one to find a bounded absorbing set and to prove asymptotic compactness of $S(t)$; The second method is to find compact absorbing set, and it turns out to be the method we employ in proving existence of random attractor in our case. The existence theorems and continuous dependence of initial datum in the earlier chapter allows us to define a flow $\varphi : \mathbb{R} \times \Omega \times H \to H$ in the following sense:

$$\varphi_t(\omega)v_0 = v(t, \omega; v_0), \quad t \in T, \omega \in \Omega, \ v_0 \in H.$$

Our goal of this chapter is to investigate the dynamical behavior of the SNSE on 2D rotating spheres with additive stable Lévy noise

$$du(t) = [-Au(t) - B(u(t), u(t)) + \mathbf{C}u(t) + f]dt + \sum_{l=1}^{m} \sigma_l dL_l(t)e_l, \quad u(0) = u_0, \tag{4.1}$$

where $A, B, \mathbf{C}$ are respectively the Stokes operator, the bilinear operator and the Coriolis operator as defined in Chapter 4, and $f \in H$, $e_1, \cdots, e_m \in H$ are the eigenfunctions of the stokes operator $A$, $\{\sigma_l\}$ is a sequence of real numbers, $L_l(t)$, $(1 \leq l \leq m)$ are mutually independent two-sided $\beta$-stable Lévy processes $u = u(t, x, \omega)$ is now a random velocity of the fluid.
Our goal in this chapter is in threefold.

- Prove (4.1) generates a RDS $\varphi$;
- Establish the existence of random attractor for (4.1);
- Establish the existence of a Feller Markov Invariant Measure supported by the random attractor.

To this end, we study the stationary ergodic solution of an Ornstein-Uhblenceck, make transformation to obtain some estimates of the solution respectively in space $H$ and $V$, then using the compact embedding of Sobolev space, we obtain the existence of compact random set which absorbs any bounded nonrandom subset of space $H$.
In section 5.2, we introduce some key terminology such as RDS, random attractors, Markov-invariant measures to study random dynamics induced by our SNSE under jump noise. In section 5.3, we prove $\varphi$ indeed defines a random dynamical system along with a driving flow $\varphi$. This claim was accomplished by first identifying a suitable canonical sample probability space for (4.1) which ensures the linear stochastic Stokes equation remains stationary. (see 5.3.1 and 5.3.2) Then via an a priori estimate for a strong solution (bounded in $V$, compact in $H$) from the earlier chapter, we identified a compact absorbing set and consequently deduce the existence of a random attractor based on the assumption of a finite-dimensional noise. Finally, using the property of random attractor, we deduce the existence of random invariant measure which is supported by the random attractor of the spherical SNSE.

## 4.2 Attractors in the theory of random dynamical system

In this section, we review the necessary mathematics foundation for the studies of random attractors. The presentation here follows closely with [7, 18, 34] with some slight modification to Jump case based on the various papers on RDS under Lévy noise [2, 57]. The notion of random dynamical system is simply a generalisation of a deterministic dynamical system. In brief, an RDS has two features, one is the measurable dynamical system $\varphi$, which is used to model the random perturbations, and the other is the cocycle mapping $\vartheta$ defined over the dynamical system (see Arnold [7] for more detail).

### 4.2.1 Basic definitions

In this subsection, we recall the definition of random dynamical system (RDS) and cocylcle

**Definition 4.2.1.** A triple $\mathfrak{T} = (\Omega, \mathcal{F}, \vartheta)$ is said to be a *measurable dynamical system* (DS) if $(\Omega, \mathcal{F})$ is a measurable space and $\vartheta : \mathbb{R} \times \Omega \ni (t, \omega) \mapsto \vartheta_t \omega \in \Omega$ is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable map such that for all $t, s \in \mathbb{R}$, $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$. A quadruple $\mathfrak{T}(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called a *metric dynamical system* (RDS) iff. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathfrak{T}' := (\Omega, \mathcal{F}, \vartheta)$ is a measurable DS such that for each $t \in \mathbb{R}$, the map $\vartheta_t : \Omega \to \Omega$ is $\mathbb{P}$-preserving.

**Definition 4.2.2.** Given a metric DS $\mathfrak{T}$ and a Polish space $(X, d)$, a map $\varphi : \mathbb{R} \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ is called a *measurable random dynamical system* (on $X$ over $\vartheta$), iff

- $\varphi(t)$ is strongly measurable $\forall\, t \in T : \varphi(t, \cdot)x$ is $\mathcal{F}/\mathcal{B}(X)$-measurable $\forall t \in T$ and $x \in X$;
- $\varphi(t, \omega)\cdot : X \to X$ is continuous for all $(t, \omega) \in \mathbb{R} \times \Omega$;
- The trajectories $\varphi(\cdot, \omega)x : \mathbb{R} \to X$ are càdlàg $\forall\, (\omega, x) \in \Omega \times \mathbb{R}$;
- $\varphi$ is $\vartheta$-cocycle:

$$\varphi(t + s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega) \quad \forall \quad s, t \in \mathbb{R},\ \varphi(0, \omega) = \mathrm{id},\ \forall\ \omega \in \Omega. \tag{4.2}$$

It follows from the cocycle property that $\varphi(t, \omega)\cdot$ is automatically invertible. ($\forall t \in T$ and $\forall\, \mathbb{P}$ a.e. $\omega$.) In fact, $\varphi(t, \omega)^{-1} = \varphi(-t, \vartheta_t \omega)$ for $t \in T$. Instead of assuming (4.2) for all $\omega \in \Omega$ it suffices to assume (4.2) for all $\omega$ from a measurable $(\vartheta_t)_{t \in T}$-invariant subset of $\Omega$ of full measure.

### 4.2.2 Stochastic Calculus for two-sided time

While we will assume our metric dynamical system has two sided time $T = \mathbb{R}$, in this subsection we briefly discuss the extension of stochastic calculus to two sided time. The material follows closely with section 2.3.2 [7].

Let $(\Omega, \mathcal{F}, P)$ denotes from now a complete probability space.

**Definition 4.2.3** (Two-Parameter Filtration, p.71 [7])**.** Assume $\mathcal{F}_s^t$, $s, t \in \mathbb{R}$, $s \leq t$, is a two parameter family of sub $\sigma$-algebras of $\mathcal{F}$ with the following properties

- $\mathcal{F}_s^t \subset \mathcal{F}_u^v$ for $u \leq s \leq t \leq v$
- $\mathcal{F}_s^{t+} := \cap_{u>t}\mathcal{F}_s^u = \mathcal{F}_s^t$, $\mathcal{F}_{s-}^t := \cap_{u<s}\mathcal{F}_u^t = \mathcal{F}_s^t$ for $s \leq t$,
- $\mathcal{F}_s^t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ for every $s \leq t$.

Then $\mathcal{F}_s^t$, $s \leq t$ is called a (two-parameter) filtration on (on $(\Omega, \mathcal{F}, P)$). We define

$$\mathcal{F}_{-\infty}^t := \vee_{s\leq t}\mathcal{F}_s^t, \quad \mathcal{F}_s^{\infty} := \vee_{t\geq s}\mathcal{F}_s^t.$$

**Definition 4.2.4** (Filtered DS, p.72 [7])**.** Let $(\Omega, \mathcal{F}^0, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ be a metric DS, let $\mathcal{F}$ be the $\mathbb{P}$-completion of $\mathcal{F}^0$, and let $\mathcal{F}_s^t$, $s \leq t$, be a filtration in $(\Omega, \mathcal{F}, \mathbb{P})$. We call $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta\}_{t\in\mathbb{R}}, \{\mathcal{F}_s^t\}_{s\leq t})$ is filtered DS, if for all $s, t, u \in \mathbb{R}$, $s \leq t$, we have

$$\vartheta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}.$$

### 4.2.3 Attraction and absorption

For two random sets $A, B \subset X$, we put

$$d(A, B) = \sup_{x\in A} d(x, B) \quad \text{and} \quad \rho(A, B) = \max\{d(A, B), d(B, A)\}.$$

In fact, $\rho$ restricted to the family $\mathfrak{C}$ of all nonempty closed subsets on $X$ is a metric (see [26]), and it is the so-called Hausdorff metric. From now on, let $\mathfrak{X}$ be the Borel $\sigma$-field on $\mathfrak{C}$ generated by open sets w.r.t. the metric $\rho$ [19, 26, 32].

**Definition 4.2.5.** Let us assume that $(\Omega, \mathcal{F})$ is a measurable space and let $(X, d)$ be a Polish space. A set-valued map $C : \Omega \to \mathfrak{C}(X)$ is said to be measurable iff. $C$ is $(\mathcal{F}, \mathfrak{X})$-measurable. Such a map is often called a *closed and bounded random set.* A closed and bounded random set $C$ will be called a *compact random set* on $X$ if for each $\omega \in \Omega$, $C(\omega)$ is a compact subset of $X$.

**Example 2.** A closed set valued map $K : \Omega \to 2^X$ is a random closed set.

**Remark.** Let $f : X \mapsto \mathbb{R}$, be a continuous function on the Polish space $X$ and $\mathbb{R} : \Omega \mapsto \mathbb{R}$ an $\mathcal{F}$-measurable random variable. If the set $C_{f,R}(\omega) := \{x : f(x) \leq R(\omega)\}$ is nonempty for each $\omega \in \Omega$, then $C_{f,R}$ is a closed and bounded random set.

**Definition 4.2.6.** Let $\varphi : \mathbb{R} \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ be measurable RDS on a Polish space $(X, d)$ over a metric $DS$ $\mathfrak{T}$. A closed random set $B$ is said to be $\varphi$ forward invariant iff. for all $\omega \in \Omega$,

$$\varphi(t, \omega)B(\omega) \subseteq B(\vartheta_t\omega) \quad \forall \quad t > 0.$$

A closed random set $B$ is said to be *strictly $\varphi$ invariant* iff. $\forall\, \omega \in \Omega$,

$$\varphi(t, \omega)B(\omega) = B(\vartheta_t\omega) \quad \forall \quad t > 0.$$

**Remark.** By substituting $\vartheta_{-t}\omega$ for $\omega$, we have the equivalent version of the above definition:

$$\varphi(t, \varphi_{-t}\omega)B(\vartheta_{-t}\omega) \subseteq B(\omega), \quad \forall\ t > 0,$$

$$\varphi(t, \varphi_{-t}\omega)B(\vartheta_{-t}\omega) = B(\omega), \quad \forall\ t > 0.$$

**Definition 4.2.7.** For a given closed random set $B$, the $\Omega$-limit set of $B$ is defined to be the set

$$\Omega(B, \omega) = \Omega_B(\omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi(t, \varphi_{-t}\omega)B(\vartheta_{-t}\omega)}$$

**Remark.** (i) A priori $\Omega(B, \omega)$ can be an empty set.

(ii) One has the following equivalent version of Definition 4.2.7:

$$\Omega_B(\omega) = \{y : \exists t_n \to \infty, \{x_n\} \subset B(\vartheta_{-t_n}\omega), \lim_{n \to \infty} \varphi(t_n, \vartheta_{-t_n}\omega)x_n = y\}.$$

(iii) Since $\overline{\bigcup_{t \geq T} \varphi(t, \vartheta_{-t}\omega)}$ is closed, $\Omega_B(\omega)$ is closed as well.

Given a probability space, a random attractor is a compact random set, invariant for the associated RDS and attracting every bounded random set in its basis of attraction. More precisely,

**Definition 4.2.8.** A random set $A : \Omega \to \mathfrak{C}(X)$ is a *random attractor* iff

- $A$ is a compact random set;

- $A$ is $\varphi$-invariant, i.e. $\mathbb{P}$-a.s.

$$\varphi(t,\omega)A(\omega) = A(\vartheta_t\omega), \tag{4.3}$$

- $A$ is attracting, in the sense that, for all $B \in X$ it holds

$$\lim_{t\to\infty} \rho(\varphi(t,\vartheta_{-t}\omega)B(\varphi_{-t}\omega), A(\omega)) = 0.$$

The random attractor $A$ in the present chapter shall not be confused with the Stokes operator $A$.
Let us now state a result on the existence of a random attractor, which is a generalisation of the Gaussian noise case in the pioneering work in [34] to the Lévy noise case.

**Theorem 4.2.9.** *Let $\varphi$ be a continuous in space, but càdlàg in time RDS on $X$. Assume the existence of a compact random set $K$ absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor $A$ given by*

$$A(\omega) = \overline{\bigcup_{B\subseteq X, B \text{ bounded}} \Omega_B(\omega)}, \quad \omega \in \Omega.$$

**Proof.** The proof is analogous to the proof of Theorem 3.11 in Flandoli and Crauel [34]. □

### 4.2.4 Invariant measures on random sets

In the final section of this thesis, we prove the existence of invariant measure for the RDS $\varphi$ (Put in another way, the existence of random invariant measure). Recall the standard facts from Chapter 2, in particular, the definition of skew product; we are ready to discuss what one means by *random invariant measure*.

**Definition 4.2.10.** Let $\varphi$ be a given RDS over a metric DS $\mathfrak{T}$. A probability measure $\mu$ on $(\Omega \times X, \mathcal{F} \times \mathcal{B})$ is said to be an invariant measure for $\varphi$ iff.

- $\Theta_t$ preserves $\mu : \Theta_t(\mu) = \mu$ for all $t > 0$;
- The first marginal of $\mu$ is $\mathbb{P}$, i.e. $\pi_\Omega(\mu) = \mathbb{P}$ where

$$\pi_\Omega : \Omega \times X \ni (\omega, x) \mapsto \omega \in \Omega.$$

The following corollary gives the existence of invariant measure for a RDS $\varphi$. The proof follows from the Markov-Katutani fixed point theorem (see p.87 [32] for more detail.).

**Corollary 6** (p.374,[34]). *Let $\varphi$ be an RDS, and suppose $\omega \mapsto A(\omega)$ is a compact measurable forward invariant set for $\varphi$. Then there exist invariant measure for $\varphi$ which are supported by A.*

Alternatively, one can construct the random invariant measure more explicitly via an Krylov-Bogoliubov type argument; we refer readers to p.87 [32].

#### 4.2.4.1 Markov Invariant Measures

Based on the conditions in Theorem 4.2.9, it is clear the attractor is measurable with respect to the past $\mathcal{F}^-$, since $\Omega_B$ is measurable for any nonrandom $B$.

Define two $\sigma$-algebra corresponding to the future and the past, respectively by

$$\mathcal{F}^+ = \sigma\{\omega \mapsto \varphi(\tau, \vartheta_t\omega) : \tau, t \geq 0\},$$

and

$$\mathcal{F}^- = \sigma\{\omega \mapsto \varphi(\tau, \vartheta_{-t}\omega) : \tau, 0 \leq \tau \leq t\}.$$

Then $\vartheta_t^{-1}\mathcal{F}^+ \subset \mathcal{F}^+$ for all $t \geq 0$ and $\vartheta_t^{-1}\mathcal{F}^- \subset \mathcal{F}^-$ for all $t \leq 0$.

**Proposition 4.2.11.** *Suppose $\omega \mapsto A(\omega)$ is an $\varphi$-invariant compact set which is measurable with respect to the past $\mathcal{F}^-$ for an RDS $\varphi$. Then there exist invariant measures $\mu$ supported by A such that also $\omega \mapsto \mu_\omega$ is measurable with respect to $\mathcal{F}^-$.*

**Corollary 7** (p.374[34]). *Under the conditions of the Proposition suppose in addition that $\varphi$ is an RDS whose one-point motions form a Markov family, and such that $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent. Then there exists an invariant measure $\rho$ for the associated Markov semigroup. Furthermore, the limit*

$$\mu_\omega = \lim_{t\to\infty} \varphi(t, \vartheta_{-t}\omega)\rho$$

*exists $\mathbb{P}$ a.s., $\rho = \int \mu_\omega d\mathbb{P}(\omega) = \mathbb{E}(\mu_\cdot)$, and $\mu$ is a Markov measure.*

#### 4.2.4.2 Feller Markov Invariant measures

By Corollary 6 for an given RDS $\varphi$ on a Polish space $X$, one can find an invariant probability measure if an invariant compact random set $K(\omega)$, $\omega \in \Omega$ can be identified. Hence Corollary 6 is generalised as the following.

**Corollary 8.** *A continuous in space, càdlàg in time RDS which has an invariant compact random set $K(\omega)$, $\omega \in \Omega$ has at least one invariant probability measure $\mu$ in the sense of definition 4.2.10.*

One of the desirable property of solutions of stochastic PDE is Feller property (see definition in Chapter 4). Let us now define a Feller invariant measure for a Markov RDS $\varphi$. If $f : X \to \mathbb{R}$ is bounded Borel measurable function, then put

$$(P_t f)(x) = \mathbb{E}f(\varphi(t, x)), \quad t \geq 0, x \in X. \tag{4.4}$$

It is clear that $P_t f$ is also a bounded and borel measurable function. Moreover, one has the following result.

**Proposition 4.2.12.** *Assume that that RDS $\phi$ is a.s. contiuous in space for every $t \geq 0$. Then the family $(P_t, t \geq 0)$ is Feller, i.e. $P_t f \in C_b(X)$ if $f \in C_b(X)$. Moreover, if the RDS $\varphi$ is càdlàg in time, then for any $f \in C_b(X)$, $(P_t f)(x) \to f(x)$ as $t \downarrow 0$.*

**Proof.** For the first part, let us fix $t > 0$. If $x_n \to x$ in $X$, then it follows from the space continuity of $\varphi(t, \cdot)$ that $(P_t f)(x_n) \to (P_t f)(x)$ using the Lebesgue dominated convergence theorem.

For the second part, note that for a given $x \in X$ from the càdlàg property of $\varphi(\cdot, x, \omega) : [0, \infty) \to X$ for a.e. $\omega$ it follows that one has $(P_t f)(x) \to f(x)$ as $t \to 0$ if $x \in X$. $\square$

A RDS $\varphi$ is called Markov iff the family $(P_t, t \geq 0)$ is a semigroup on $C_b(X)$, that is, $P_{t+s} = P_t \circ P_s$ for all $t, s \geq 0$.

**Definition 4.2.13.** A Borel probability measure $\mu$ in $H$ is said to be invariant w.r.t. $P_t$ if

$$P_t^* \mu := \int_X P_t(x, \Gamma)\mu(dx) = \mu(\Gamma), \quad \forall \Gamma \in \mathcal{B}(X),$$

where $(P_t^*)(\Gamma) = \int_H P_t(x, \Gamma)\mu(dx)$ for $\Gamma \in \mathcal{B}(H)$ and $P_t(x, \cdot)$ is the transition probability, $P_t(x, \Gamma) = P_t(1_\Gamma)(x)$

Finally, a Feller invariant probability measure for a Markov RDS $\varphi$ on $H$ is, by definition, an invariant probability measure for the semigroup $(P_t, t \geq 0)$ define by (4.4).

In view of Corollary 7, if a Markov RDS $\varphi$ on a Polish space $H$ has an invariant compact random set $K(\omega), \omega \in \Omega$, then there exists a Feller invariant probability measure $\mu$ for $\varphi$. More precisely we have the following corollary.

**Corollary 9.** *If a càdlàg time and space continuous RDS $\varphi$ has an invariant compact random set, then there exists a feller invariant probability measure $\mu$ for $\varphi$.*

## 4.3 Random Dynamical systems generated by the SNSE on a rotating unit sphere

Having established the well-posedness in the earlier chapter, we are in a position to define an RDS $\varphi$ corresponding to the problem (3.61) in $H$. But first, we need to determine a sample (canonical) probability space for which the dynamics of the driving noise remains stationary.

### 4.3.1 Some analytic facts

Recall that $X = \mathbb{L}^4(\mathbb{S}^2) \cap H$ denote the Banach space endowed with the norm

$$|x|_X = |x|_H + |x|_{\mathbb{L}^4(\mathbb{S}^2)}.$$

Recall Assumption 1 last chapter, namely, the space $K \subset H \cap \mathbb{L}^4$ is a Hilbert space such that for any $\delta \in (0, 1/2)$,

$$A^{-\delta} : K \to H \cap \mathbb{L}^4(\mathbb{S}^2) \quad \text{is } \gamma\text{-radonifying.} \tag{4.5}$$

This assumption is satisfied if $K = D(A^s)$ for some $s > 0$.

**Remark.** Under the above assumption the space $K$ can be taken as the RKHS of the cylindrical Wiener process $W(t)$ on $H \cap \mathbb{L}^4$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where $\Omega = D(\mathbb{R}, E)$ of càdlàg functions defined on $\mathbb{R}$ take value in $E$ with the following Skorohod metric

$$d(l_1, l_2) = \sum_{i=1}^{\infty} (1 \wedge d_i^\circ(l_1, l_2)) \quad \forall\ l_1, l_2 \in D,$$

where $l_1^i(t) := g_i(t) l_1(t)$ and $l_2^i(t) := g_i(t) l_2(t)$ with

$$g_i(t) := \begin{cases} 1, & \text{if } |t| \le i - 1 \\ i - t, & \text{if } i - 1 \le |t| \le i \\ 0, & \text{if } |t| \le i \end{cases}$$

$$d_i^{\circ}(l_1^i, l_2^i) = \inf_{\lambda \in \Lambda} \left( \sup_{-i \le s < t \le i} |\log \frac{\lambda(t) - \lambda(s)}{t - s}| \vee \sup_{-i \le t \le i} |l_1(t) - l_2(\lambda(t))| \right),$$

where $\Lambda$ denotes the set of strictly increasing, continuous function $\lambda(t)$ from $\mathbb{R}$ to itself with $\lambda(0) = 0$. This skorohod space is a complete separable metric space which is taken as the canonical sample space. Let $\mathcal{F}$ be the Borel $\sigma$-algebra of the Polish space $(D(\mathbb{R}, X), d)$. For every $t \in \mathbb{R}$ we have the evaluation map $L_t : D(\mathbb{R}, X) \to \mathbb{R}$ denote by $L_t(\omega) = \omega(t)$. Then we have $\mathcal{F} = \sigma(L_t, t \in \mathbb{R})$, that is, $\mathcal{F}$ is the smallest $\sigma$-algebra generated by the family of maps $\{L_t : t \in \mathbb{R}\}$. Let $\mathbb{P}$ be the unique probability measure which makes the canonical process a two-sided Lévy process (see definition in Chapter 3) with paths in $D(\mathbb{R}; E)$, that is, $\omega(t) = L_t(\omega)$. Define the shift

$$(\vartheta_t \omega)(\cdot) = \omega(t + \cdot) - \omega(t) \quad t \in \mathbb{R}, \omega \in \Omega.$$

Then the map $(t, \omega) \to \vartheta_t(\omega)$ is continuous and measurable [7] and the (Lévy) probability measure $\mathbb{P}$ is $\vartheta$ invariant, that is, $\mathbb{P}(\vartheta_t^{-1}(T)) = \mathbb{P}(T)$ for all $T \in \mathcal{F}$. This flow is an ergodic dynamical system with respect to $\mathbb{P}$. Thus $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta)_{t \in \mathbb{R}})$ is a metric DS.

### 4.3.2 Ornstein-Uhlenbeck process

In the following subsections we are concerned with the linear evolutionary Stokes equations. In particular the notations $H$ and $A$ are as defined in subsection 3.1.3. The space $X$ is defined in subsection 4.3.1.
Recall, the equation

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), & t \in [0, T], \\ u(0) = u_0. \end{cases}$$

If $A$ generates a $C_0$-semigroup in a Banach space $E$ and $f : [0, T] \to \mathbb{E}$ is such a function that

$$\int_0^T |f(t)|_E \, dt < \infty,$$

then the solution is given by

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s) ds.$$

In particular, we have

**Proposition 4.3.1.** *Let $L$ be a Lévy process taking value in $E$, such that for any $T > 0$*

$$\int_0^T |L(t)|_E \, dt < \infty .$$

*Then the solution of the differential equation*

$$\dot{V}(t) + \alpha V(t) = L(t), \quad V(0) = 0, \quad \alpha > 0$$

*is given by*

$$V(t) = \int_0^t e^{-(t-s)A} L(s) ds.$$

#### 4.3.2.1 Stochastic convolution and integrating by parts

Here we quote a useful integration by part formula from [114] which allows us to attain the desired regularity for which the RDS $\varphi$ exist.
Consider the following Ornstein-Uhlenbeck process generated by the Stokes operator on $\mathbb{S}^2$,

$$z_t = \int_0^t e^{-A(t-s)} G dL(s) = \sum_{l=1}^{\infty} z_l e_l,$$

where $\{e_l : l = 1, \cdots\}$ is the complete orthonormal system of eigenfunctions of $A$ in $H$ and

$$z_t(t) = \int_0^t e^{-\lambda_l(t-s)} \sigma_l dL^l(s), \tag{4.6}$$

where $\lambda_l$ are the eigenvalues of the Stokes operator $A$. By the Itô product formula, see Theorem 4.4.13 of [4] for any $l \geq 1$, one has that

$$L^l(t) = \int_0^t \lambda_l e^{-\lambda_l(t-s)} L^l(s) ds + \int_0^t e^{-\lambda_l(t-s)} dL^l(s) + \int_0^t \lambda_l e^{-\lambda_l(t-s)} \Delta L^l(s) ds,$$

where $\Delta L^l(s) = L^l(s) - L^l(s-)$. Since $L^l(t)$ is a $\beta$ stable process, $\Delta L^l(s) = 0$ a.e. for $s \in [0, t]$ and so we have

$$\int_0^t \lambda_l e^{-\lambda_l(t-s)} \Delta L^l(s) ds = 0.$$

Therefore,

$$z_l(t) = \sigma_l L^l(t) - \int_0^t \lambda_l e^{-\lambda_l(t-s)} \sigma_l L^l(s) ds.$$

Hence, if we assume that $\sigma_l = 0$ for $l > m$ for a certain finite $m > 1$ then

$$z(t) = L(t) - Y(t),$$

where

$$Y(t) = \int_0^t A e^{-A(t-s)} L(s) ds.$$

In this case we clearly have

$$Y(t) \in H \quad \text{a.s.} \quad (t \geq 0).$$

#### 4.3.2.2 Regularity of Shifting flow

To prove our stochastic Navier-Stokes system generates a RDS, we will transform it into a random PDE in $X$ with aid of the integration by part technique introduced earlier. We need to give a meaning to the Ornstein-Uhlenbeck process in the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ given by

$$z(\vartheta_t\omega) := \hat{z}(t) = \int_{-\infty}^{t} \widehat{A}^{1+\delta} e^{-(t-r)\widehat{A}}(\tilde{\omega}(t) - \tilde{\omega}(r))dr, \quad t \in \mathbb{R}.$$

Our goal now is to show $\widehat{z}(t)$ is a well defined element in $X := \mathbb{L}^4(\mathbb{S}^2) \cap H$ for a.e. $\omega$. But first, we need the following couple of results, which can be viewed as generalisation of Theorem 4.1 and Theorem 4.4 in [23] to the case where the Ornstein Uhlenbeck generator is $\hat{A} + \alpha I$ in place of $A$, where

$$\widehat{A} = \nu A + C, \quad D(\hat{A}) = D(A), \quad A = -P(\boldsymbol{\Delta} + 2\mathrm{Ric}).$$

Recall that

$$|\widehat{A}^{1+\delta} e^{-t(\hat{A}+\alpha I)}|_{\mathcal{L}(X,X)} \leq Ct^{-1-\delta} e^{-(\mu+\alpha)t}, \quad t > 0. \tag{4.7}$$

**Proposition 4.3.2.** *Assume $\beta \in (1,2)$, $p \in (0,\beta)$ and*

$$\sum_{l=1}^{\infty} |\sigma_l|^{\beta} \lambda_l^{\beta/2} < \infty\,.$$

*Then*

$$\mathbb{E} \int_{-\infty}^{t} |\hat{A} e^{-(t-r)\hat{A}}(\tilde{\omega}(t) - \tilde{\omega}(r))|_X^p dr < \infty. \tag{4.8}$$

*Moreover, for* $\mathbb{P}$ almost every $\tilde{\omega} \in D(\mathbb{R}, X)$, $t \in \mathbb{R}$ *the function*

$$\hat{z}(t) = \hat{z}(\tilde{\omega})(t) = \int_{-\infty}^{t} \hat{A} e^{-(t-r)\hat{A}}(\tilde{\omega}(t) - \tilde{\omega}(r))dr, \quad t \in \mathbb{R} \tag{4.9}$$

*is well defined and cádlág in X. Furthermore, for any $\kappa > 0$ such that $\kappa p > 1$ there exists a random variable C depending on $\beta$, p, $\sigma$, $\delta$ such that*

$$|\hat{z}(\tilde{\omega})(t)|_X \leq C(\beta, p, \sigma, \delta, \tilde{\omega})(1 + |t|^{\kappa}). \tag{4.10}$$

**Proof.** *Part I* We will show first that the Lévy process $L$ is is càdlàg in $X$ and

$$\mathbb{E} \sup_{t\leq T} |L(s)|_X^p < \infty\,. \tag{4.11}$$

By Lemma 3.2.4 the process $L$ is cádlág in $V$, hence in $\mathbb{L}^2\left(\mathbb{S}^2\right)$ and finally in $X$. It remains to show (4.11). Recall Lemma 3.2.1, that is,

$$\mathbb{E}\sup_{t\leq T}|A^{\delta}L(t)|^p \leq C(\beta,p)\left(\sum_{l\geq 1}|\sigma_l|^{\beta}\lambda_l^{\beta\delta}\right)^{\frac{p}{\beta}} t^{\frac{p}{\beta}} < \infty. \tag{4.12}$$

Putting $\delta = 0$, we have,

$$\mathbb{E}\sup_{t\leq T}|L(t)|^p \leq C(\beta,p)\left(\sum_{l\geq 1}|\sigma_l|^{\beta}\right)^{\frac{p}{\beta}} t^{\frac{p}{\beta}} < \infty, \tag{4.13}$$

and putting $\delta = \frac{1}{2}$ we have,

$$\mathbb{E}\sup_{t\leq T}|A^{\frac{1}{2}}L(t)|^p \leq C(\beta,p)\left(\sum_{l\geq 1}|\sigma_l|^{\beta}\lambda_l^{\beta/2}\right)^{\frac{p}{\beta}} t^{\frac{p}{\beta}} < \infty. \tag{4.14}$$

So

$$\begin{aligned}
\mathbb{E}\sup_{t\leq T}|L(s)|^p_{\mathrm{X}} &\leq c\mathbb{E}\sup_{t\leq T}|L(s)|^p + c\mathbb{E}\sup_{t\leq T}|L(s)|^p_{\mathbb{L}^4(\mathbb{S}^2)} \\
&\leq c\mathbb{E}\sup_{t\leq T}|L(s)|^p + C\mathbb{E}\sup_{t\leq T}\left(|L(s)|^{\frac{p}{2}}|L(s)|_V^{\frac{p}{2}}\right) \quad \text{via Ladyzhenskaya inequality} \\
&\leq c\mathbb{E}\sup_{t\leq T}|L(s)|^p + C\mathbb{E}\sup_{t\leq T}|L(s)|^p_V \quad \text{via Lemma 3.23} \\
&= C(\beta,p)\left(\sum_{l\geq 1}|\sigma_l|^{\beta}\right)^{p/\beta} s^{\frac{p}{\beta}} + C(\beta,p)\left(\sum_{l\geq 1}|\sigma_l|^{\beta}\lambda_l^{\beta/2}\right)^{\frac{p}{\beta}} s^{\frac{p}{\beta}} \\
&\leq C(\beta,p)\left(\sum_{l\geq 1}|\sigma_l|^{\beta}\lambda_l^{\beta/2}\right)^{\frac{p}{\beta}} s^{\frac{p}{\beta}}.
\end{aligned}$$

*Part II* In what follows we use the fact that $\tilde{\omega}(t) = L(t)$ $\mathbb{P}$-a.s. Using the change of variables $s = t - r$, we obtain

$$\begin{aligned}
\mathbb{E}\int_{-\infty}^{t}|\hat{A}e^{-(t-r)\hat{A}}(\tilde{\omega}(t)-\tilde{\omega}(r))|^p dr &= \int_0^{\infty}\mathbb{E}|\hat{A}e^{-s\hat{A}}(\tilde{\omega}(t)-\tilde{\omega}(t-s))|^p_{\mathrm{X}} ds \\
&= \int_0^{\infty}\mathbb{E}|\hat{A}e^{-s\hat{A}}\tilde{\omega}(s)|^p_{\mathrm{X}} ds.
\end{aligned}$$

Using (4.7) with $\gamma = \alpha + \mu$ we have

$$\int_0^\infty \mathbb{E}|\hat{A}e^{-s\hat{A}}\tilde{\omega}(s)|_X^p ds \leq C\int_0^\infty \frac{e^{-p\gamma s}}{s^p}\mathbb{E}|A^\delta\tilde{\omega}(s)|_X^p ds$$
$$\leq C\int_0^\infty \frac{e^{-p\gamma s}}{s^p}C(\beta,p)s^{\frac{p}{\beta}}(\sum_{l\geq 1}|\sigma_l|^\beta\lambda_l^{\beta\delta})^{\frac{p}{\beta}}ds < \infty,$$

since $p - \frac{p}{\beta} < 1$ and we infer that $\widehat{z}(t)$ is well defined in $X$ $\mathbb{P}$-a.s. using the same arguments as in the proof of (4.11) above.

We will prove (4.10). Applying Lemma 3.5.9, with the Banach space $B = X$ and $\kappa$ such that $\kappa p > 1$ we obtain

$$|\widehat{z}(\tilde{\omega})(t)|_X \leq C(\beta,p,\sigma,\delta,\kappa,\tilde{\omega})(1+|t|^\kappa), \tag{4.15}$$

and (4.10) follows.

*Part III* One has to check $\hat{z}$ is right continuous with left limit in $X$. To this end note first that

$$\begin{aligned}\widehat{z}(t) &= \int_{-\infty}^t \widehat{A}e^{-(t-s)\hat{A}}\,(\omega(t)-\omega(s))\,ds\\ &= \left(\widehat{A}\int_0^\infty e^{-s\hat{A}}ds\right)\omega(t) - \int_{-\infty}^t \widehat{A}e^{-(t-s)\hat{A}}\omega(s)\,ds\\ &= \omega(t) - \int_{-\infty}^t \widehat{A}e^{-(t-s)\hat{A}}\omega(s)\,ds,\end{aligned}$$

since $\widehat{A}$ is invertible. The function $\omega$ is càdlàg in $X$ by assumption. We will show that the function

$$F(t,\omega) = \int_{-\infty}^t \widehat{A}e^{-(t-s)\hat{A}}\omega(s)\,ds$$

is continuous in $X$ for $\mathbb{P}$-a.e. $\omega$. Indeed, for $s,t\in\mathbb{R}$ such $r<t$ we have

$$\begin{aligned}\int_{-\infty}^t \widehat{A}e^{-(t-s)\widehat{A}}\omega(s)\,ds &= \int_{-\infty}^r \widehat{A}e^{-(t-s)\widehat{A}}\omega(s)\,ds + \int_r^t \widehat{A}e^{-(t-s)\widehat{A}}\omega(s)\,ds\\ &= \widehat{A}e^{-(t-r)\widehat{A}}\int_{-\infty}^r e^{-(r-s)\widehat{A}}\omega(s)\,ds + \int_r^t \widehat{A}e^{-(t-s)\widehat{A}}\omega(s)\,ds\\ &= \widehat{A}e^{-(t-r)\widehat{A}}h + I(t)\,.\end{aligned}$$

Since the semigroup $e^{-s\widehat{A}}$ is analytic, we find that the function $t\to\widehat{A}e^{-(t-r)\widehat{A}}h$ is continuous for $t>r$. Let us consider $I(t)$. By Sobolev embeddings we have a continuous embedding $\mathbb{H}^{1,2}\subset\mathbb{W}^{\frac{1}{4},4}$. Therefore for $\delta$ small enough the function $t\to A^\delta\omega(t)$ is locally bounded in $\mathbb{L}^4$ for a.e. $\omega$. Then

$$I(t) = \int_r^t \widehat{A}^{1-\delta}e^{-(t-s)\widehat{A}}\widehat{A}^\delta\omega(s)\,ds$$

is continuous for $t>r$, again by standard properties of analytic semigroups.

□

**Theorem 4.3.3.** *Under the assumption of Proposition 4.3.2, for $\mathbb{P}$-a.e. $\omega \in D(\mathbb{R}, X)$,*

$$\hat{z}(\vartheta_s\omega)(t) = \hat{z}(\omega)(t+s), \quad t, s \in \mathbb{R}.$$

**Proof.** The proofs of the first three parts follows from closely from Theorem 4.8 and Proposition 8.4 in [23], see also Theorem 9 in [63]. For the last part, since $(\vartheta_s\omega)(r) = \omega(r+s) - \omega(s)$, $r \in \mathbb{R}$, we have

$$\begin{aligned}
\hat{z}(\vartheta_s\omega)(t) &= \int_{-\infty}^{t} Ae^{-(t-r)A}[\vartheta_s\omega(t) - \vartheta_s\omega(r)]dr \\
&= \int_{-\infty}^{t} \widehat{A}e^{-(t-r)A}[\omega(t+s) - \omega(r+s)]dr \\
&= \int_{-\infty}^{t+s} \widehat{A}e^{-(t+s-r')A}[\omega(t+s) - \omega(r')]dr' = \hat{z}(\omega)(t+s).
\end{aligned}$$

□

Now, put $(\tau_s\zeta)(t) = \zeta(t+s)$, $t, s \in \mathbb{R}$. Therefore $\tau_s$ is linear, bounded map from $D(\mathbb{R}, X)$ into $D(\mathbb{R}, X)$. Moreover, the family $(\tau_s)_{s\in\mathbb{R}}$ is a $C_0$ group on $D(\mathbb{R}, X)$. Hence the shifting property could be re-expressed as

**Corollary 10.** *For $\mathbb{P}$-a.e. $\omega \in D(\mathbb{R}, X)$ For $s \in \mathbb{R}$, $\tau_s \circ \hat{z} = \hat{z} \circ \vartheta_s$, that is*

$$\tau_s(\hat{z}(\omega)) = \hat{z}(\vartheta_s(\omega)), \quad \omega \in D(\mathbb{R}; X).$$

**Proposition 4.3.4.** *The process*

$$z_\alpha(t) = \int_{-\infty}^{t} e^{-(t-s)(\hat{A}+\alpha I)}dL(s),$$

*where the integral is intepreted in the sense of [23] is well defined and is identified as a solution to the equation*

$$dz_\alpha(t) + (\hat{A} + \alpha I)z_\alpha dt = dL(t), \quad t \in \mathbb{R}.$$

*The process $z_\alpha$, $t \in \mathbb{R}$ is a stationary OU process.*
We define

$$z_\alpha(\omega) := \hat{z}(\hat{A} + \alpha I; \omega) \in D(\mathbb{R}, X),$$

i.e. for any $t \geq 0$,

$$z_\alpha(\omega)(t) := \int_{-\infty}^{t} (\hat{A} + \alpha I)e^{-(t-r)(\hat{A}+\alpha I)}(\omega(t) - \omega(s))ds$$

By Proposition 4.3.1,

$$\frac{d^+}{dt}z_\alpha(\omega)(t) + (\hat{A}+\alpha I)\int_{-\infty}^{t}(\hat{A}+\alpha I)^{1+\delta}e^{-(t-r)(\hat{A}+\alpha I)}(\omega(t)-\omega(s))ds = L(t).$$

Therefore $z_\alpha(t)$ satisfies

$$\frac{d^+}{dt}z_\alpha(t) = (\hat{A}+\alpha I)z_\alpha = \dot{\omega}(t), t \in \mathbb{R}. \tag{4.16}$$

It follows from Theorem 4.3.3 that

$$z_\alpha(\vartheta_s\omega)(t) = z_\alpha(\omega)(t+s), \quad \omega \in D(\mathbb{R}, X),\ t, s \in \mathbb{R}.$$

Similar to our definition of Lévy process $L_t$, i.e. $L_t(\omega) := \omega(t)$, we can view the ODE as a definition of $z_\alpha(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, equation (4.16) suggests that this process is an Ornstein Uhblenck process.

Now we have enough tools to prove the cocycle property of RDS, and this allows us to prove $(\varphi, \vartheta)$ is an RDS. The proof follows same lines as Theorem 6.15 in [18].

### 4.3.3 Random dynamical system generated by the SNSE on a sphere with Lévy noise

Let us fix $\alpha \geq 0$ and put $\Omega = \Omega(E)$.

We define a map $\varphi = \varphi_\alpha : \mathbb{R} \times \Omega \times H \to H$ by

$$(t, \omega, x) \mapsto v(t, \hat{z}_\alpha(\omega))(x - \hat{z}_\alpha(\omega)(0)) + \hat{z}_\alpha(\omega)(t).$$

In what follows, write $z = z_\alpha$ for simplicity.

Put in another way,

$$\begin{aligned}\varphi = \varphi_\alpha(t,\omega)x &:= v(t, z_\alpha(\omega))(x - z_\alpha(\omega)(0)) + z_\alpha(\omega)(t)\\ &= u(t,\omega;x) \quad \forall\ t \in T, \omega \in \Omega,\ x \in H,\end{aligned}$$

where $u(\cdot;\omega,u_0)$ is the solution of the integral equation corresponding to given $\omega \in \Omega$, $u_0 \in H$ and $\varphi$ satisfies the definition of RDS.

Since $\varphi(t) = \varphi(t, \vartheta_t(\omega))v_0$ and $v(0) = v_0$. Then $\varphi(0,\omega) = I$. It is clear that $\varphi(0,\omega) = I$. Because $z(\omega) \in D(\mathbb{R}; X)$, $z(\omega)(0)$ is a well-defined element of $H$ and hence $\varphi$ is well defined. Furthermore, we have the first main result of this chapter.

**Theorem 4.3.5.** *$(\varphi, \vartheta)$ is a random dynamical system.*

To prove the claim, one simply check the definition of a random dynamical system (see subsection 4.2.2).

**Proof.** First, we check *Measurability*. Suppose $u_0 \in V$ and $t \in T$ is fixed, the map $\omega \mapsto \varphi(t,\omega)u_0 \in H$ is measurable because the solution $u(t,\omega;u_0)$ is constructed as the (poinwise in $\omega$) limit of sucessive approximation of the contraction, which is measurable being explicitly defined in term of measurable objects. Finally, if $u_0 \in H$, then $u(t,\omega;u_0)$ is the limit of $u(t,\omega;u_n^0)$ with $u_n^0 \in V$. The required measurability is assured.
Next, we check *Continuous dependence on initial data*.
The proof follows the similar line of the proof of uniqueness in subsubsection 3.4.2.
Third, we check *Càdlàg property of* $\varphi(t,\omega)$. This turns out to be an easy task: The càdlàg property of $\varphi_t u_0$ is clear from the proof of existence and uniqueness of the solution $u$.
Lastly, we will check the *cocycle property* of $\varphi$, namely, for any $x \in H$, one has to check,

$$\varphi(t+s,\omega)x = \varphi(t,\vartheta_s\omega)\varphi(s,\omega), \quad t,s \in \mathbb{R}. \tag{4.17}$$

From the definition of $\varphi$, noting from the cocycle property of $z$, $\hat{z}(\vartheta_s\omega)(t) = \hat{z}(\omega)(t+s)$, $z(\omega)(s) = z(\vartheta_s\omega)(0)$ for all $s \in \mathbb{R}$, we have for all $t,s \in \mathbb{R}$,

$$\varphi(t+s,\omega)x = v(t+s, z(\omega)(t+s))(x - z(\omega)(0)) + z(\omega)(t+s),$$

$$\begin{aligned}\varphi(t,\vartheta_s\omega)\varphi(s,\omega)x &= v(t, z(\vartheta_s\omega)(t))(x - z(\omega)(0)) + z(\omega)(s) - z(\vartheta_s\omega)(0) + z(\vartheta_s\omega)(t)\\ &= v(t, z(\vartheta_s\omega)(t))(v(s, z(\omega))(s))(x - z(\omega)(0)) + z(\vartheta_s\omega)(t).\end{aligned}$$

In view of (4.3.3), to prove (4.17), we need to prove

$$v(t+s, z(\omega)(t+s))(x - z(\omega)(0)) = v(t, z(\vartheta_s\omega)(t))(v(s, z(\omega)(0))).$$

Now, fix $s \in \mathbb{R}$, define $v_1$, $v_2$ by

$$\begin{aligned}v_1(t) &= v(t+s, z(\omega)(t+s))(x - z(\omega)(0)) \quad t \in \mathbb{R},\\ v_2(t) &= v(t, z(\vartheta_s\omega)(t))(v(s, z(\omega)(s))(x - z(\omega)(0))), \quad t \in \mathbb{R}.\end{aligned}$$

Because $v(0, z(\vartheta_s\omega)(0))(x - z(\vartheta_s\omega)(0)) = x - z(\vartheta_s\omega)(0)$, one infer that

$$\begin{aligned}v_1(0) &= v(s, z(\omega)(s))(x - z(\omega)(0))\\ &= v(0, z(\vartheta_s\omega)(0))(v(s, z(\omega)(s))(x - z(\omega)(0))) = v_2(0).\end{aligned}$$

Since $\mathbb{R} \ni t \mapsto v(t, z(\omega))$ is a solution to

$$\begin{cases}\frac{dv}{dt+} = -\nu Av - B(v) - B(v,z) - B(z,v) - B(z) + \alpha z + f,\\ v(0) = v_0.\end{cases}$$

On the other hand, in view of our earlier existence uniqueness results, the fact $v$ takes value in $D(A)$ implies that $v(t)$ is differentiable for almost every $t$. We have

$$\begin{aligned} v'(t) &= -\nu A v_1(t+s, z(\omega)(t+s)) - B(v_1(t+s, z(\omega)(t+s)) + z(\omega)(t+s) + \alpha z(\omega)(t+s) + f \\ &= -\nu A v_1(t, z(\omega)) - B(v_1(t, z(\omega)) + z(\omega)(t+s)) + \alpha z(\omega)(t+s) + f. \end{aligned}$$

On the other hand for $v_2$,

$$\frac{dv(t, z(\vartheta_s\omega)(t))}{dt+} = -\nu A v(t, z(\vartheta_s\omega)(t)) - B(v(t, z(\vartheta_s\omega)(t))) + z(\vartheta_s\omega)(t) + \alpha z(\vartheta_s\omega)(t) + f.$$

Therefore, $v_1$, $v_2$ solve respectively

$$\begin{cases} v_1'(t) = -\nu A v_1' - B(v_1'(t) + z(\omega)(t+s)) + \alpha z(\omega)(t+s) + f, \\ v_1(0) = v(z(\omega))(s)(x - z(\omega)(0)), \end{cases}$$

$$\begin{cases} v_2'(t) = -\nu A v_2' - B(v_2'(t) + z(\vartheta_s\omega)(t)) + \alpha z(\vartheta_s\omega)(t) + f, \\ v_1(0) = v(z(\omega))(s)(x - z(\omega)(0)). \end{cases}$$

By cocycle property of $z$, $z(\vartheta_s\omega)(t) = z(\omega)(t+s)$ for $t \in \mathbb{R}$. □

Therefore, $v_1$, $v_2$ are solutions to (3.75) with the same initial data $v(s, z(\omega)(s))(x - z(\omega)(0))$ at $t = 0$. Then it follows from the uniqueness of solution to (3.75) that $v_1 = v_2$, $t \in \mathbb{R}$.

### 4.3.4 Existence of random attractors

We have the Poincare inequalities

$$|u|_V^2 \geq \lambda_1 |u|^2, \quad \forall \quad u \in V, \tag{4.18}$$

$$|Au|^2 \geq \lambda_1 |u|^2, \quad \forall \quad u \in D(A). \tag{4.19}$$

**Lemma 4.3.6.** *Suppose that $v$ is a solution to problem (3.75) on the time interval $[t_0, \infty)$ with $z \in L^4_{loc}(\mathbb{R}, \mathbb{L}^4(\mathbb{S}^2)) \cap L^2_{loc}(\mathbb{R}, V')$ and $t_0 \geq 0$. Then, for any $t \geq \tau \geq t_0$, one has*

$$|v(t)|^2 \leq |v(\tau)|^2 e^{\int_\tau^t \gamma(s)ds} + \int_{t_0}^t e^{\int_s^t \gamma(\xi)d\xi} 2p(s)ds, \tag{4.20}$$

*where*

$$p(t) = c|f|^2 + c\alpha|z|^2 + \delta|z|^2 \sum_{l=1}^m |z_l(t)|, \tag{4.21}$$

$$\gamma(t) = -\frac{\lambda_1}{2} + 4\beta \sum_{l=1}^{m} |z_l(t)| \tag{4.22}$$

*for all* $t_0 \leq \tau \leq t$ *and* $c$ *depends only on* $\lambda_1$

**Proof.** The proof will be provided shortly. □

The main lines of proving the existence of random attractors follow from classical lines of proving Global attractors by finding compact absorbing sets. However, as pointed out in the paper [34], the analysis of Navier-Stokes equations with additive noise in our case requires some non-trivial consideration. In particular, a critical question arised when analyzing the estimate $\frac{d^+}{dt}|v(t)|^2$, the usual estimates for the nonlinear term $b(v(t), z(t), v(t))$ yields a term $|v(t)|^2|z(t)|_4^4$, so we were not able to deduce any bound in $H$ for $|v(t)|^2$ under the classical lines (see for instance section 6 in [14]). Nevertheless, in light of the method developed in [34], via the usual change of variable and by writing the noise and the Ornstein-Uhlenbeck process as an infinite sequence of 1D processes, we are able to show there exist random attractors to our system 1.27 as well. In what proceed we will detail our proof.

Let $H$, $A : D(A) \subset H \to H$, $V = D(A^{1/2}) = D(\hat{A}^{1/2})$ and $B(u, v) : V \times V \to V'$, $Cu$ be spaces and operators introduced in the previous section. Suppose that there exists a constant $c_B > 0$ such that

$$\langle B(u, v), w\rangle = |b(u, v, w)| \leq c_B|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2}|w|_V, \quad \forall \quad u, v, z \in V, \tag{4.23}$$

$$\langle B(u, v), v\rangle \leq c_B|u|^{1/2}|Au|^{1/2}|v|_V|z|$$

for all $u \in D(A)$, $v \in V$ and $z \in H$. Moreover, let $f \in H$, $e_1, \cdots, e_m \in H$ be given, $\{\sigma_l\}$ is a sequence of real numbers. Consider 4.1 again,

$$du(t) = [-Au(t) - B(u(t), u(t)) + Cu(t) + f]dt + \sum_{l=1}^{m} \sigma_l dL_l(t) e_l, \quad u(0) = u_0.$$

As in last chapter, assume that $e_l$ are the eigenfunctions of the stoke operator $A$, $1 \leq l \leq m$, there exists $\delta > 0$ such that

$$|\langle B(u, e_l), u\rangle| \leq \delta|u|^2, \quad u \in V, l = 1, \cdots, m. \tag{4.24}$$

**Remark.** In bounded domain or in $\mathbb{S}^2$, one has

$$\langle B(u, e_l), u\rangle = \sum_{i,j=1}^{3} \int_{\mathbb{S}^2} u_i \frac{\partial (e_l)_j}{\partial x_i} u_j dx. \tag{4.25}$$

In this case assumption (4.24) is satisfied when $e_l$ are Lipschitz continuous in $\mathbb{S}^2$. Put $L(t) = \sum_{l=1}^{m} e_l L_l(t)$.

#### 4.3.4.1 Stochastic flow

Consider the abstract SNSE

$$du + [Au + B(u) + Cu]dt = fdt + GdL(t),$$

and the Ornstein-Uhlenback equation

$$dz + (\hat{A} + \alpha I)zdt = GdL(t).$$

From the discussion from the earlier subsubsection, it is clear that $z(t)$ is a stationary ergodic solution with continuous trajectories take value in $D(A)$. So we can transform the SNSE to a random PDE. The main advantage is that we can solve the equation $\omega$-wise due to the absence of the stochastic integral.

We now use the change of variable $v(t) = u(t) - z(t)$. Then, by subtracting the Ornstein-Uhlenback equation from the abstract SNSE, we find that $v$ satisfies the following equation

$$\frac{dv}{dt+} = -\nu Av(t) - Cv(t) - B(u, u) + f + \alpha z. \tag{4.26}$$

Now recall Theorem 3.4.5 from last chapter.

**Theorem 4.3.7.** *Assume that equation (3.190) is satisfied. Then for $\mathbb{P}$-a.s. $\omega \in \Omega$, there hold*

- *For all $t_0 \in \mathbb{R}$ and all $v_0 \in H$, there exists a unique solution $v \in C([t_0, +\infty]; H) \cap L^2_{loc}([t_0, +\infty); V)$ of equation (4.26) with initial value $v_0$.*
- *If $v_0 \in V$, then the solution belongs to $C([t_0, +\infty); V) \cap L^2_{loc}([t_0, +\infty); D(A))$.*
- *hence, for every $\varepsilon > 0$, $v(t) \in C([t_0 + \varepsilon, +\infty); V) \cap L^2_{loc}([t_0 + \varepsilon, +\infty); D(A))$.*
- *Denoting the solution by $v(t, t_0; \omega, v_0)$, then the map $v_0 \mapsto v(t, t_0; \omega, v_0)$ is continuous for all $t \geq t_0$, $v_0 \in H$.*

Now Let us define the transition semigroup for the flow $\varphi$ as

$$P_t f(x) = \mathbb{E} f(\varphi(t, x)).$$

**Corollary 11.** *It follow from Theorem 3.4.5 the transition semigroup for the Markov RDS $\varphi$ has Feller property in $H$. That is, $P_t : C_b(H) \to C_b(H)$*

Having the map $v_0 \mapsto v(t, t_0; \omega, v_0)$, where $v(t, t_0; \omega, v_0)$ is the solution to (4.26) with $v(t_0) = v_0$, we can now define a stochastic flow $\varphi(t, \omega)$ in $H$ by setting

$$\varphi(t, \omega)u_0 = v(t, 0; \omega, u_0 - z_\alpha(\omega)(0)) + z_\alpha(\omega)(t).$$

#### 4.3.4.2 Absorbing in $H$ at time $t = -1$

In what proceed, assume $\omega \in \Omega$ is fixed; the results will hold $\mathbb{P}$ a.s.. Suppose $t_0 \le -1$ and $u_0 \in H$ be given, and let $v$ be the solution of equation (4.26) for $t \ge t_0$, with $v(t_0) = u_0 - z(t_0, \omega)$ (which was denoted above by $v(t, 0; \omega, u_0 - z(0, \omega))$). Using the well known identity $\frac{1}{2}\partial_t |v(t)|^2 = (v(t), v(t))$, and the assumption $\langle B(u, v), v\rangle = 0$ and the antisymmetric term $(Cv, v) = 0$ we have

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 = -\nu(Av, v) - \langle B(u, z), u\rangle + (\alpha z, v) + \langle f, v\rangle \tag{4.27}$$

$$\le -\nu|v|_V^2 - \langle B(u, z), u\rangle + \alpha|z||v| + |f||v|. \tag{4.28}$$

By the definition of $z$ and assumptions (3.190),

$$|\langle B(u, z), u\rangle| = \left|\sum_{l=1}^{m}\langle B(u, e_l), u\rangle e_l\right| \le \delta|u|^2\sum_{l=1}^{m}|z_l|$$

$$\le 2\delta|v|^2\sum_{l=1}^{m}|z_l| + 2\delta|z|^2\sum_{l=1}^{m}|z_l|.$$

and the inequalities

$$(\alpha z, v) = c\alpha|z|^2 + c'|v|^2,$$

$$\langle f, v\rangle \le c|f|^2 + c'|v|^2.$$

For simplicity we take $\nu = 1$. Then via Young inequality, one can show that there exists $c, c' > 0$ depending only on $\lambda_1$ such that

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \le \left(-\frac{\lambda_1}{4} + 2\delta\sum_{l=1}^{m}|z_l|\right)|v|^2 + c|f|^2 + c\alpha|z|^2 + 2c|z|_V^2 + 2\delta|z|^2\sum_{l=1}^{m}|z_l|.$$

Let $\gamma(t)$, and $p(t)$ are defined as:

$$p(t) = c|f|^2 + c\alpha|z|^2 + \delta|z|^2\sum_{k=1}^{m}|z_k(t)|,$$

$$\gamma(t) = -\frac{\lambda_1}{2} + 4\delta\sum_{l=1}^{m}|z_l(s)|,$$

we have

$$\frac{1}{2}\frac{d^+}{dt}|v|^2 + \frac{1}{2}|v|_V^2 \le \frac{1}{2}\gamma(t)|v|^2 + p(t), \tag{4.29}$$

$$\frac{d^+}{dt}|v(t)|^2 \leq \gamma(t)|v(t)|^2 + 2p(t).$$

Invoking Gronwall Lemma over the interval $[a, \infty)$, we have (4.20).

**Lemma 4.3.8.** *There exists a random radius $r_1(\omega) > 0$ such that for all $\rho > 0$ there exists (a deterministic) $\bar{t} \leq -1$ such that the following holds $\mathbb{P}$-a.s. For all $t_0 \leq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leq \rho$, the solution $v(t, t_0; \omega, u_0 - z(s))$ of equation (3.75) over $[t_0, \infty]$ with $v(t_0) = u_0 - z_\alpha(t_0)$ satisfies the inequality*

$$|v(-1, t_0; \omega, u_0 - z_\alpha(t_0, \omega))|^2 \leq r_1^2(\omega).$$

**Proof.** Apply Lemma 4.3.6 with $t = -1$, $\tau = t_0$, we have

$$\begin{aligned} |v(-1)|^2 &\leq |v(t_0)|^2 e^{\int_{t_0}^{-1} \gamma(\xi) d\xi} + \int_{t_0}^{-1} e^{\int_s^t \gamma(\xi) d\xi} 2p(s) ds \\ &\leq 2e^{\int_{t_0}^{-1} \gamma(\xi) d\xi} |u_0|^2 + 2e^{\int_{t_0}^{-1} \gamma(\xi) d\xi} |z(t_0)|^2 + \int_{-\infty}^{-1} e^{\int_s^t \gamma(\xi) d\xi} 2p(s) ds. \end{aligned} \tag{4.30}$$

Put

$$r_1^2(\omega) = 2 + 2 \sup_{t_0 \leq -1} e^{\int_{t_0}^{-1} \gamma(\xi) d\xi} |z(t_0)|^2 + \int_{-\infty}^{-1} e^{\int_s^t \gamma(\xi) d\xi} 2p(s) ds \tag{4.31}$$

which is finite $\mathbb{P}$ a.s. due to (3.185) and (3.186).
So, given $\rho > 0$, choose $\bar{t}$ such that

$$e^{\int_{t_0}^{-1} \gamma(\xi) d\xi} \rho^2 \leq 1$$

for all $t_0 \leq \bar{t}$. The claim then follows from (4.20). We remark that $t_0$ depending on $\omega$. ☐

Taking $t \in [-1, 0]$ and $\tau = -1$ in (4.20) we have

$$|v(t)|^2 \leq |v(-1)|^2 e^{\int_{-1}^t \gamma(\xi) d\xi} + \int_{-1}^t e^{\int_s^t \gamma(\xi) d\xi} 2p(s) ds.$$

Let us now come back to (4.29):

$$\frac{d^+}{dt}|v|^2 + |v|_V^2 \leq \gamma(t)|v|^2 + 2p(t).$$

Integrate over $[-1, 0]$,

$$|v(0)|^2 - |v(-1)|^2 + \int_{-1}^0 |v(s)|_V^2 ds \leq \left( \int_{-1}^0 \gamma(\xi) d\xi \right) \left( \sup_{-1 \leq t \leq 0} |v(t)|^2 \right) + \int_{-1}^0 2p(s) ds.$$

Therefore,

$$\int_{-1}^{0} |v(s)|_V^2 ds \leq |v(-1)|^2 + \left(\int_{-1}^{0} \gamma(\xi) d\xi\right) \left(\sup_{-1\leq t\leq 0} |v(t)|^2\right) + \int_{-1}^{0} 2p(s) ds.$$

Therefore, from above lemma we deduce

**Lemma 4.3.9.** *There exists two random variables $c_1(\omega)$ and $c_2(\omega)$ depending on $\lambda_1$, $e_1, \cdots, e_m$ and $|f|$ such that for all $\rho > 0$ there exists $\bar{t}(\omega) \leq -1$ such that the following holds $\mathbb{P}$ a.s. $\forall\, t_0 \leq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leq \rho$, the solution $v(t, \omega; t_0, u_0 - z(t_0, \omega))$ of equation (4.26) over $[t_0, \infty]$ with $v(t_0) = u_0 - z(t_0)$ satisfies*

$$|v(t, \omega; t_0, u_0 - z(t_0, \omega))|^2 \leq c_1(\omega) \quad \forall \quad t \in [-1, 0],$$

$$\int_{-1}^{0} |v(s, \omega; t_0, u_0 - z(t_0, \omega))|_V^2 ds \leq c_2(\omega).$$

**Proof.** Put

$$c_1(\omega) = e^{\int_{-1}^{t} \gamma(\xi) d\xi} r_1^2(\omega) + \int_{-1}^{t} e^{\int_{s}^{t} \gamma(\xi) d\xi} p(s) ds,$$

$$c_2(\omega) = r_1^2(\omega) \left(1 + \int_{-1}^{0} \gamma(\xi) d\xi\right) + \int_{-1}^{0} 2p(s) ds,$$

with $r_1(\omega)$ as in the previous lemma. Then, given $\rho > 0$, it suffices to choose $t(\omega)$ as in the proof of that previous lemma. □

#### 4.3.4.3 Absorption in $V$ at $t = 0$

From (3.75) we have (by multiplying $Av$ left and right and noting $(v_t, Av) = \frac{1}{2}\frac{d^+}{dt}|v|_V^2)$, using inequality (4.23), and use the Young inequality with $ab = \sqrt{\frac{1}{e}}a\sqrt{e}b$, $p = 2$

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2},$$

With the choice of $e = \frac{\nu}{4}$, one has that

$$\langle f, Av\rangle \leq \frac{2}{\nu}|f|^2 + \frac{\nu}{8}|Av|^2,$$

$$\langle \alpha z, Av\rangle \leq \frac{2}{\nu}|\alpha z|^2 + \frac{\nu}{8}|Av|^2,$$

$$c_B|u|^{1/2}|Au|^{1/2}|u|_V|Av| \leq 2\nu c_B^2|u||Au||u|_V^2 + \frac{\nu}{8}|Av|^2,$$

$$\begin{aligned}
\frac{1}{2}\frac{d^+}{dt}|v|_V^2 &= -\nu|Av|^2 - \langle B(u,u), Av\rangle + \langle f, Av\rangle + \langle \alpha z, Av\rangle \\
&\leq -\nu|Av|^2 + c_B|u|^{1/2}|Au|^{1/2}|u|_V|Av| + |f||Av| + |\alpha z||Av| \\
&\leq -\frac{5\nu}{8}|Av|^2 + 2\nu c_B^2|u||Au||u|_V^2 + \frac{2}{\nu}(|f|^2 + |\alpha z|^2)
\end{aligned}$$

With $q(t) = \frac{2}{\nu}(|f|^2 + |\alpha z|^2)$ and noticing $|Au| \leq |Av| + |Az|$,

$$\leq -\frac{5\nu}{8}|Av|^2 + 2\nu c_B^2|u||Av||u|_V^2 + 2\nu c_B^2|u||Az||u|_V^2 + q(t)$$

Apply Young inequality with $e = \nu/2$ for the nonlinear term,

$$\begin{aligned}
&\leq -\frac{5\nu}{8}|Av|^2 + \frac{\nu}{8}|Av|^2 + 4\nu c_B^4|u|^2|u|_V^2|u|_V^2 + 2c_B^2|u||Az||u|_V^2 + q(t) \\
&\leq -\frac{\nu}{2}|Av|^2 + 4\nu c_B^4|u|^2|u|_V^2|u|_V^2 + 2c_B^2|u||Az||u|_V^2 + q(t) \\
&\leq -\frac{\nu}{2}|Av|^2 + 8\nu c_B^4|u|^2|u|_V^2|v|_V^2 + 8\nu c_B^4|u|^2|u|_V^2|z|_V^2 + 2\nu c_B^2|u||Az||u|_V^2 + q(t).
\end{aligned}$$

Temporarily disregard the $|Av|^2$ term, we have

$$\frac{d^+}{dt}|v|_V^2 \leq 16\nu c_B^4|u|^2|u|_V^2|v|_V^2 + 16\nu c_B^4|u|^2|u|_V^2|z|_V^2 + 4\nu c_B^2|u||Az||u|_V^2 + 2q(t).$$

Invoking Gronwall Lemma, we get for $s \in [-1, 0]$,

$$|v(0)|_V^2 \leq e^{\int_s^0 16\nu c_B^4|u(\tau)|^2|u(\tau)|_V^2 ds} \times \left(|v(s)|^2 + \int_s^0 (16\nu c_B^4|u|^2|u|_V^2|z|_V^2 + 4\nu c_B^4|u||Az||u|_V^2 + 2q(t))d\sigma\right).$$

Integrating in $s$ over $[-1, 0]$ we obtain

$$|v(0)|_V^2 \leq \left(\int_{-1}^0 |v(\tau)|^2 d\tau + \int_{-1}^0 [16\nu c_B^4|u|^2|u|_V^2|z|_V^2 + 4\nu c_B^4|u||Az||u|_V^2 + 2q(t)]d\sigma\right) e^{\int_{-1}^0 16\nu c_B^4|u(\tau)|^2|u(\tau)|_V^2 d\tau}. \tag{4.32}$$

**Lemma 4.3.10.** *There exists a random radius $r_2(\omega) > 0$, depending only on $\lambda_1$, $e_1, \cdots, e_m$ and $f$, such that for all $\rho > 0$ there exists (a deterministic) $\bar{t} \leq -1$ such that the following holds $\mathbb{P}$-a.s. For all $t_0 \leq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leq \rho$, the solution $v(t, t_0, \omega; u_0 - z(t_0, \omega))$ of equation 4.26 over $[t_0, \infty]$ with $v(t_0) = u_0 - z_\alpha(t_0)$, put $u(t, t_0, \omega; u_0) = z(t, \omega) + v(t, t_0, \omega; u_0 - z(t_0, \omega))$. Then*

$$|u(0, \omega; t_0, u_0 - z_\alpha(t_0, \omega))|_V^2 \leq r_2^2(\omega).$$

**Proof.** In view of (4.32), we need to estimate the term.

$$\int_{-1}^{0} |u(s)|^2 |u(s)|_V^2 ds.$$

Now using the fact $u = v + z$ and $|u|^2 \le 2|v|^2 + 2|z|^2 = 2(|v|^2 + |z|^2)$,, then the two terms can be estimated as following.

$$\begin{aligned}
&\int_{-1}^{0} |u(s)|^2 |u(s)|_V^2 ds \\
&\le \sup_{-1\le t\le 0} |u(t)|^2 \int_{-1}^{0} |u|_V^2 ds \\
&\le \sup_{-1\le t\le 0} 2(|v(t)|^2 + |z(t)|^2) \left( \int_{-1}^{0} 2|v(\tau)|^2 + 2|z(\tau)|^2 \right) \\
&\le 2(c_1(\omega) + \sup_{-1\le t\le 0} |z(t)|^2) 2(c_2(\omega) + \int_{-1}^{0} |z(s)|^2 ds) \\
&= 2c_3(\omega) 2c_4(\omega),
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^{0} |u(s)||u(s)|_V^2 ds \\
&\le \sup_{-1\le t\le 0} |u(t)| \int_{-1}^{0} |u(s)|_V^2 ds \\
&\le (c_1(\omega) + \sup_{-1\le t\le 0} |z(t)|) 2c_4(\omega).
\end{aligned}$$

Hence, put

$$\begin{aligned}
c_3(\omega) &= c_1(\omega) + \sup_{-1\le t\le 0} |z(t)|^2, \\
c_4(\omega) &= c_2(\omega) + \int_{-1}^{0} |z(t)|_V^2 ds, \\
c_5(\omega) &= c_1(\omega)^{1/2} + \sup_{-1\le t\le 0} |z(t)|.
\end{aligned}$$

Then, (4.32) becomes

$$\begin{aligned}
|u(0)|_V^2 &\le 2|z(0)|_V^2 + 2|v(0)|_V^2 \\
&\le 2|z(0)|_V^2 \\
&+ 2\left[ c_2(\omega) + 64c_B^4 c_3(\omega)c_4(\omega) \sup_{-1\le t\le 0} |z(t)| + 8c_B^2 c_5(\omega)c_4(\omega) \sup_{-1\le t\le 0} |Az(t)| + \int_{-1}^{0} 2q(s)ds \right] e^{64c_B^4 c_3(\omega)c_4(\omega)} \\
&=: r_2^2(\omega).
\end{aligned}$$

□

Hence there exists a random ball in $V$ which absorbs the bounded sets of $H$. Since $V$ is compactly embedded in $H$, there exists a compact set $K \subset H$ such that, for all bounded set $B \subset H$ there exists $\bar{t} \leq -1$ such that $\varphi B \subset K$ $\mathbb{P}$ almost surely.

### 4.3.5 Existence of Feller Markov Invariant Measures

In this subsection, we prove the existence of random attractor implies the existence Feller Markov invariant measures.

**Theorem 4.3.11.** *The stochastic flow associated with the SNSE with additive Lévy noise (4.1) has a compact random attractor, in the sense of Theorem 4.2.9. Moreover, the Markov semigroup induced by the flow of $H$ has an invariant measure $\rho$ in the sense of Corollary 7. The associated flow-invariant Markov measure $\mu$ on $H \times \Omega$ has the property that its disintegration $\omega \mapsto \mu_\omega$ is supported by the attractor.*

**Proof.** Recall that, in the language of the stochastic flow associated with our SNSE (3.72),

$$u(0, \omega; t_0, u_0) = \varphi(t_n, \vartheta_{-t_0}\omega)u_0 = v(0, \omega; t_0, u_0 - z(s)) + z(t).$$

Then by the previous lemma, there exists a random ball in $V$ which absorbs the bounded sets of $H$. Since $V$ is compactly embedded in $H$, there exists a compact set $K \subset H$ such that, for all bounded set $B \subset H$ there exists $\bar{t} \leq -1$ such that $\varphi B \subset K$ $\mathbb{P}$ a.s.. Defining $K(\omega) := \{u \in H : |u| \leq r_2(\omega)\}$, we have proved the existence of a compact absorbing set. Then by Theorem 4.2.9, there exists random attractor to (4.1). The existence of an invariant Markov measure is a direct consequence of Corollary 9, provided we can show that the one-point motions associated with the flow $\varphi(t, \omega)$ define a family of Markov processes. The proof of this is analogous to the proof of Markov property of solutions to the (4.1) in the last chapter. Nevertheless we repeat here as well. Let $\varphi_{s,t}$ be defined as in earlier section. Let $\mathcal{F}_{s,t}$ be the $\sigma$-algebra generated by $L(r) - L(s)$ for all $r \in [s, t]$, and let $\mathcal{F}_t = \mathcal{F}_{0,t}$. Define the operators $P_t$ in the space of bounded measurable function over $H$ as $(P_t f)(u_0) = \mathbb{E}f(\varphi(t)u_0)$. To prove $\varphi(t, \omega)$ defines a family of Markov processes. It suffices to prove

$$\mathbb{E}[f(\varphi(t + s)x)|\mathcal{F}_t] = P_s(f)(\varphi(t)x),$$

for all $0 \leq s \leq t$ and all bounded continuous functions $f$ over $H$, which implies that $\varphi(t + s)x$ is a Markov process with transition semigroup $P_t$. By uniqueness, the following holds

$$\varphi(t + s, \omega)x = \varphi(s, \omega)\varphi(t, \omega)x$$

over $[t,\infty]$ with $\mathcal{F}_t$ measurable initial condition $\varphi(t,t)\delta = \delta$. It suffices to prove

$$\mathbb{E}[f(\varphi_{t,t+s}\delta)|\mathcal{F}_t] = P_s(f)(\delta) \tag{4.33}$$

for for every $H$ integrable, $\mathcal{F}_t$ r.v. $\delta$.
Note, (4.33) not only holds for every $f \in C_b(H)$, but also holds for $\varphi = 1_\Gamma$, where $\Gamma$ is an arbitrary Borel set of $H$ and consequently for all $\varphi \in B_b(H)$. Without loss of generality, we assume $\varphi \in C_b(H)$. We know that, if $\delta = \delta_i$ $\mathbb{P}$ a.s., then $r.v.\, \varphi(t,t+s)\delta_i$ is independent to $\mathcal{F}_t$, since $\varphi(t,t+s)\delta_i$ is $\mathcal{F}_{t,t+s}$ measurable. Hence,

$$\mathbb{E}(f(\varphi(t,t+s)\delta_i)|\mathcal{F}_t) = \mathbb{E}f(\varphi(t,t+s)\delta_i) = P_{t,t+s}f(\delta_i) = P_sf(\delta_i), \quad \mathbb{P} \quad \text{a.s.} \quad .$$

Since the coefficient of the equation for $\varphi(t,t+s)$ are independent, one can see that the $H$ r.v. $\varphi_{t,t+s}$ and $\varphi_s x$ have the same law. If $\delta$ has the form

$$\delta = \sum_{i=1}^{N} \delta^i 1_{\Gamma^i}, \tag{4.34}$$

where $\delta^{(i)} \in H$ and $\Gamma^{(i)} \subset \mathcal{F}_t$ is a partition of $\Omega$, $\delta_i$ are elements of $H$. Then

$$\varphi(t,t+s)\delta_i = \sum_{i=1}^{N} \varphi(t,t+s,\delta_i)1_{\Gamma_i}, \quad \mathbb{P}, \quad \text{a.s.} \quad .$$

Hence,

$$\mathbb{E}(f(\varphi(t,t+s)\delta)|\mathcal{F}_t) = \sum_{i=1}^{N} \mathbb{E}(f(\varphi(t,t+s)\delta_i)1_{\Gamma_i}|\mathcal{F}_t) \quad \mathbb{P} \quad \text{a.s.} \quad .$$

Take into account the r.v. $u(t,t+s)\delta_i$ independent to $\mathcal{F}_t$ and $1_{\Gamma_i}$ are $\mathcal{F}_t$ measurable, $i = 1, \cdots, l$, one deduces that

$$\mathbb{E}[f(\varphi(t,t+s,\delta))|\mathcal{F}_t] = \sum_{i=1}^{N} P_sf(\delta_i)1_{\Gamma_i} = P_sf(\delta), \quad \mathbb{P} \quad \text{a.s.}$$

and so (4.33) is proved. For a general $\delta$ there exists a sequence of $\delta_n$ for which (4.33) holds converges to $\delta$ in $L^2(\Omega; H)$ a.s., that is,

$$\mathbb{E}|\delta - \delta_n|^2 \to 0.$$

By continuity of $f$ one can pass in the identity (4.33), with $\delta$ replaced with $\delta_n$, to the limit and (4.33) holds if $\mathbb{E}|\delta|^2 < \infty$. So $\varphi(t,\omega)$ defines a family of Markov processes.
The proof of existence of Markov measure is completed. □

**Remark.** Although the same results hold in $\beta$-stable Lévy case as in the Gaussian case (see [34]), there is some difference between dealing with Brownian motion and Lévy motions. First, we need to consider càdlàg function in the Skorohod metric, which are different from the continuous case in the metric under the compact-open topology. Second, one has to consider solutions in the sense of Carthéodory and the right-hand derivatives.

Let $u(t, x)$ be the unique solution to problem (3.72). Let us recall from last chapter that such a unique solution exist for each $x \in H$. Let us define the transition operator $P_t$ by a standard formula. For $f \in C_b(H)$, put

$$(P_t f)(x) = \mathbb{E} f(\varphi(t, x)), \quad t \geq 0, x \in X.$$

In view of Proposition 4.2.12, $(P_t, t \geq 0)$ is a family of Feller operators, i.e. $P_t : C_b(H) \to C_b(H)$ and, for any $f \in C_b(H)$ and $x \in H$, $P_t f(x) \to f(x)$ as $t \downarrow 0$. Moreover, following the identical lines of the proof of Theorem 4.3.11 in last subsection, one can prove that $\varphi$ is a Markov RDS. Invoke corollary 9, we deduce the existence of Feller invariant measure for our stochastic Navier-Stokes equations (3.72) or (4.1).

**Corollary 12.** *There exists an Feller Markov Invariant Measure for the SNSE (4.1)*

# CHAPTER 5

## Concluding Remarks

In this thesis, we extend the stochastic analysis initiated by Goldys et al. [14, 15] to the case where noise is taken as a Lévy process of stable type. The approach we have taken is functional analytic in the sense that the governing stochastic Navier-Stokes equation is interpreted as an ordinary stochastic differential equation in a Hilbert space $H$.

The core parts of this thesis are chapters 2-4. Chapter 2 concerns with the stochastic analysis of the Lévy process. We derive and prove a new version of stochastic Fubini theorem for stochastic integral w.r.t. stable white noise. In Chapter 3, we prove, under suitable conditions of the noise term that the existence and uniqueness of the solution in a fixed probability space based on a priori estimates and some classical PDE arguments. Specifically, for the weak solution, we use the classical Galerkin approximation and a compactness argument, and for the strong solution, a fixed point argument is used. Then, in the second part of Chapter 3, we deduce the existence of an invariant measure. The question of uniqueness of invariant measure in our problem remained an open question. One must point out that this problem is difficult and there have been no results for the Lévy noise of stable type, even the 2D bounded domain case. A brief reason is that there is a trade-off between well-posedness of the SNSE and the strong Feller property (see the publication [42]). Nevertheless, the ergodicity results for the SNSE driven by Lévy noise with a non-degenerate Gaussian term is developed in [40]. The notion of asymptotic strong Feller developed by Hairer and Mattingly [61] may also be helpful to tackle the question of ergodicity in our case.

In Chapter 4, a canonical sample space is identified for the SNSE for which the stochastic Stokes equations remain stationary. Much effort is spent on finding the conditions to ensure a well defined Ornstein-Uhlenbeck which possesses the shifting property. Then combined this with the well-posedness result in Chapter 4, it is shown that the SNSE defines a RDS $\varphi$. In the second part of Chapter 4, we establish the existence of a global (random) attractor which carries a random invariant measure, under the assumption of finite dimensional noise. One possible future direction from our work is to find the Hausdorff dimension of the random attractor $\dim(A)$. The primary motivation for estimating Hausdorff dimension (for both PDE and SPDE) is to show that, the dynamics of the system settles down to a finite-dimensional

object in an asymptotic regime despite the phase space is infinite dimensional. Given this, the dynamics of an infinite dimensional system reduces down to a finite dimensional object. Thus, obtaining sharp estimates of this dimension is extremely useful in the study the asymptotic regime of an infinite dimensional system, in the sense of allowing us to anticipate the minimum number of degree of freedom that a reduced system must possess to capture the essential feature of the asymptotic dynamics. The problem of estimating Hausdorff dimension is very challenging since the random attractor is essentially a random set which is not uniformly bounded. Flandoli and Crauel [35] developed a method to estimate Hausdorff dimensions for the 2D NSE with bounded noise. To overcome the lack of uniform boundedness, a very restrictive assumption has to be used. Moreover, the method does not work for the more general form of equations. From a future perspective, it perhaps worthwhile to develop an efficient reduction theory of SPDE, see the new book [27].
In addition to what was mentioned above. Here is the list of a possible future direction of this study we have in mind.

*The Numerical analysis of SNSE on the sphere with either Gaussian or Lévy noise*
The numerical study of SNSE plays an important role in the real-world modelling of turbulent fluids. Numerical methods have been developed for deterministic NSE on the sphere (see for instance Fengler and Freeden [48]). The numerical implementation of SNSE on the sphere is an open area of research even in the Gaussian case.

*SNSE on the compact Riemannian manifold with Lévy noise.*
The sphere is the simplest kind of Riemannian manifold, hence extending to a general Riemannian manifold is a natural extension of the work done in this thesis. We believe studying SNSE on manifold would be interesting from a pure mathematics point of view, especially for researchers working at the cross-section of Differential Geometry, PDE theory and stochastic analysis.

# Abbreviations and Notations

**a.e.** ........... Almost Everywhere
**a.s.** ............ Almost Surely
$B_b$ ............ Bounded Borel Measurable function
$C_b$ ............ Bounded continuous function
$C_0^\infty$ ............ Infinite differentiable function with compact support
$D(A)$ .......... Domain of operator $A$
**DS** ............ Dynamical System
**i.i.d.** ........... Independent Identically Distributed random variable
$\mathcal{L}$ ............. Probability Law
$L(U, H)$ ....... The space of bounded linear operators from $U$ into $H$
$R(\mathcal{H}, U)$ ....... The set of all $\gamma$-radonifying operators from $\mathcal{H}$ into $U$
**RDS** .......... Random Dynamical System
**ODE** .......... Ordinary Differential Equations
**OU** ........... OrnsteinUhlenbeck
**PDE** .......... Partial Differential Equations
**RKHS** ........ Reproducing Kernel Hilbert Space
**r.v.** ............ Random variable
**RCLL** ......... Right continuous with left limit
**SNSE** ......... Stochastic Navier-Stokes Equations
**SPDE** ......... Stochastic Partial Differential Equations
**spt** ............ support

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